Hedging Basket Credit Derivative Claims: A Local Risk-Minimization Approach

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June 29, 2005
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Motivation

- Basket credit derivatives have seen an exponential growth over the last few years especially with the emergence of standard indices such as iTraxx and CDX.

- The main credit derivative structures are: Credit Default Swaps, First-To-Default Swaps, Collateralized Debt Obligations.

- FTDs and CDOs are correlation products, their value depends on the likelihood of multiple defaults. They are usually hedged with single-name CDS.

- Correlation risk introduces a market incompleteness. To address this problem, we use a local-risk minimization approach.

- Bi-modal nature of credit distributions means that we need to hedge spread risk and default risk.
Credit Derivatives

- **Credit Default Swap**: is a bilateral agreement whereby the protection buyer makes a series of premium payments until the maturity of the trade or default, depending on which occurs first. In return, he receives a protection against the default of the reference entity. In the event of default, the protection seller makes a payment equal to the difference between the face value of the obligation and its recovery value after default.

- **First-To-Default Swap**: is a default swap where the protection seller takes on exposure to the first entity suffering a credit event within a basket. Similarly, we can have second-to-default swaps, third-to-default swaps, and so forth.

- **CDO (or first loss swap)**: is a default swap where the protection seller commits to cover all the losses incurred on a portfolio within a pre-defined range. In return, the protection buyer pays a periodic premium on a notional that amortizes with losses on the portfolio.


**Definition 1** A copula function is the multivariate distribution function of \( n \) random variables uniformly distributed on \([0, 1]\).

**Theorem 2** (Sklar (1959)) For \( n \) random variables \((X_1, ..., X_n)\) with marginal distribution functions \((F_1, ..., F_n)\) and joint distribution function \(F\), there exists an \( n \)-dimensional copula \(C\) such that for all \((x_1, x_2, ..., x_n) \in \mathbb{R}^n\),

\[
F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n))
\]

if \(F_1, F_2, ..., F_n\) are continuous, then \(C\) is unique.

- **Examples:**
  - Gaussian Copula / T-Copula.
  - Marshall-Olkin Copula.
Marshall-Olkin Copula

- The Marshall-Olkin copula was traditionally used in reliability theory to model the failure rate in multi-component systems. The failure of each component is assumed to be contingent on some independent Poisson shocks.

- This is also known as a multivariate Poisson model or a Poisson shock model.

- It was first used in the context of basket credit derivatives pricing in Duffie (1998), then in Duffie and Garleanu (2001).

- It has a number of useful analytical results for the aggregate portfolio distribution (see Lindskog and McNeil (2003)).
Quadratic Hedging Approaches

- **Complete Market**: perfect replication by self-financing strategies.

- **Incomplete Market**:
  - **Local Risk Minimization**: allow mean self-financing strategies and minimize the “Expected Additional Cost”
    \[
    r_t(\varphi) = \mathbb{E} \left[ \left( C_{t+1}(\varphi) - C_t(\varphi) \right)^2 \mid G_t \right],
    \]
    The value process of a local risk-minimizing strategy is given by
    \[
    V_t^H = \mathbb{E} [H \mid G_t],
    \]
    where \( \mathbb{E} [\cdot \mid G_t] \) is the conditional expectation operator under the **Minimal Martingale Measure** \( \hat{P} \).
  - **Mean Variance Hedging**: keep the self-financing property and give-up the perfect replication
    find \( V_0 \in \mathbb{R} \) and \( \alpha \in \Theta \) such that \( \mathbb{E} \left[ (V_0 + G_T(\alpha) - H)^2 \right] \) is minimal.
The Model (1)

We work in an economy represented by a probability space \((\Omega, \mathcal{G}, P)\) and a time horizon \(T^* \in (0, \infty)\), on which is given a \(d\)-dimensional Brownian motion \(W\), and a set of \(n\) non-negative random variables \((\tau_1, \ldots, \tau_n)\) representing the default times of the obligors in the economy.

We introduce an \(\mathbb{R}^d\)-valued Itô process \(X_t\), describing the evolution of the state-variables in the economy, which solves the following SDE

\[
dX_t = \alpha(X_t)\,dt + \beta(X_t)\,dW_t,
\]

for some continuous functions \(\alpha_k : \mathbb{R}^d \to \mathbb{R}\) and \(\beta_{kj} : \mathbb{R}^d \to \mathbb{R}^d\), \(1 \leq k \leq d, 1 \leq j \leq d\).

We denote by \(\{\mathcal{F}_t\}_{0 \leq t \leq T^*}\) the filtration generated by \(X\) and augmented with the \(P\)-null sets of \(\mathcal{G}\):

\[
\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t) \vee \mathcal{N}.
\]
The Model (2)

We introduce, for each obligor $i$, the right-continuous process $D_t^i \triangleq 1_{\{\tau_i \leq t\}}$ indicating whether the firm has defaulted or not. We denote by $\{\mathcal{H}_t^i\}$ the filtration generated by this process

$$\mathcal{H}_t^i \triangleq \mathcal{F}_t^{D_t^i} = \sigma \left( D_s^i : 0 \leq s \leq t \right).$$

The agents’ filtration is the one generated by the economic state variables and the default processes

$$G_t \triangleq \mathcal{F}_t \vee \left[ \bigvee_{i=1}^n \mathcal{H}_t^i \right].$$

**Assumption.** We assume that the default times are correlated and we allow for multiple instantaneous joint defaults. The multivariate dependence is defined by a Marshall-Olkin copula.
There exists a set of $m$ independent Cox processes $N_{t}^{c_{j}}$ with continuous bounded intensities $\lambda^{c_{j}}(X_{t})$, which can trigger simultaneous defaults.

Conditional on a trigger event of type $c_{j}$, at time $\theta_{r}^{c_{j}}$, we draw an independent Bernoulli variable $A_{r}^{i,j}$ with a given probability $p^{i,j} \in [0,1]$, indicating if obligor $i$ has defaulted or not.

The process $N_{t}^{i}$ defined as

$$N_{t}^{i} \triangleq \sum_{j=1}^{m} \sum_{\theta_{r}^{c_{j}} \leq t} A_{r}^{i,j},$$

is also a Cox process with intensity

$$\lambda^{i}(X_{t}) = \sum_{j=1}^{m} p^{i,j} \lambda^{c_{j}}(X_{t}).$$
The default time $\tau_i$ is defined as the first jump time of the Cox process $N_t^i$:

$$\tau_i = \inf \{ t : N_t^i > 0 \}.$$ 

This can be formally described by the following SDE

$$dD_t^i = (1 - D_t^i) \sum_{j=1}^{m} A_{t}^{i,j} dN_t^{c_j}.$$ 

The Marshall-Olkin filtration is much larger than the one accessible to the agents. It contains the evolution of the common trigger events and the “conditional” Bernoulli events:

$$\tilde{G}_t = \mathcal{F}_t \vee \left[ \bigvee_{j=1}^{m} \mathcal{F}_t^{N_{c_j}} \right] \vee \left[ \bigvee_{j=1}^{m} \bigvee_{i=1}^{n} \mathcal{F}_t^{A_{t}^{i,j}} \right].$$
In our economy, we assume that we have \((n + 1)\) primary assets available for hedging with price processes \(S^i_t = \left(S^i_t\right)_{0 \leq t \leq T^*}\).

The first asset \(S^0\) is the money-market account, i.e., \(S^0_t = \exp\left(\int_0^t r_s ds\right)\). It will be used as numeraire, and all quantities will be expressed in units of \(S^0\).

We shall consider zero-coupon credit derivatives or contingent claims of the European type.

**The hedging assets.** \(S^i\) will represent the zero-coupon defaultable bond maturing at \(T\) linked to obligor \(i\); i.e., it pays 1 if obligor \(i\) survives until time \(T\), or 0 otherwise. The payoff at maturity is defined as:

\[
S^i_T \triangleq 1 - D^i_T.
\]
In practice, zero-coupon defaultable bonds are not traded in the market. They can, however, be extracted from the prices of liquid default swap instruments with different maturities.

**Definition.** A contingent claim is a $\mathcal{G}_T$-measurable random variable $H_T$ describing the payoff at maturity $T$ of a financial instrument.

**Example 1.** The payoff of a $k^{th}$-to-default (zero-coupon note) maturing at $T$ is defined as:

$$
H_T^{(k)} \triangleq 1_{\left\{ \sum_{i=1}^n D_i T < k \right\}},
$$

it will pay 1 if there are less than $k$ defaults in the basket or 0 otherwise. The most common structure in this category is a first-to-default, $H_T^{(1)}$, which pays 1 if no obligor in the basket defaults before $T$. 


Example 2. Assuming that the recovery rate for obligor $i$ is a constant proportion $R^i \in [0, 1)$, the payoff of a CDO (zero-coupon note) covering the portfolio losses, which fall in some range $[K_1, K_2]$, where $0 \leq K_1 < K_2 \leq 1$, is

$$H^{(K_1,K_2)}_T = \frac{1}{K_2 - K_1} \max \left( \min \left( \sum_{i=1}^{n} \left(1 - R^i\right) D^i_T - K_1, 0 \right), K_2 - K_1 \right).$$

We shall consider the problem of pricing and hedging zero-coupon contingent claims by dynamically trading the hedging assets $S$. The contingent claims, in this context, include credit derivatives of the basket type.
The Problem (4)

As shown in Föllmer and Schweizer (1991), for non-attainable claims, a locally risk-minimizing strategy is characterized by: (a) its cost process must be a martingale, (b) the cost process is orthogonal to $M^S$ the martingale part of the price process $S$.

This is equivalent to having the following decomposition

$$H_T = H_0 + \int_{0,t}^1 (\alpha_t^{HT})^{tr} dS_t + L_t^{HT},$$

where $L_t^{HT}$ is a martingale orthogonal to $M^S$. This is known as the Föllmer-Schweizer decomposition.

Our goal is to find an analytical result for $(\alpha_t^{HT})$. 
The Equivalent Fatal Shock Model (1)

We use the “Equivalent Fatal Shock Model” (see Lindskog and McNeil (2003)) as a tool to equivalently describe the Marshall-Olkin model. This provides an explicit representation of the marked point process, which will be used throughout.

Let \( \Pi_n \) be the set of all subsets of \( \{1, \ldots, n\} \). For each \( \pi \in \Pi_n \), we introduce the point process \( N^\pi_t \), which counts the number of shocks in \( (0, t] \) resulting in joint defaults of the obligors in \( \pi \) only:

\[
N^\pi_t = \sum_{j=1}^{m} \sum_{r=1}^{c_j} A^{\pi,j}_{c_j} A^i_{\theta r},
\]

where

\[
A^{\pi,j}_{c_j} = \prod_{i \in \pi} A^i_{t} \prod_{i \notin \pi} (1 - A^i_{t}).
\]

We have the key result of the fatal shock representation (see Proposition 4 in Lindskog and McNeil (2003)).
The Equivalent Fatal Shock Model (2)

Proposition 3 (Fatal shock representation). \((N^\pi)_{\pi \in \Pi_n}\) are independent Cox processes, with intensities

\[
\lambda^\pi (X_t) = \sum_{j=1}^m p^{\pi,j} \lambda^{c_j} (X_t),
\]

where \(p^{\pi,j} = \prod_{i \in \pi} p^{i,j} \prod_{i \notin \pi} (1 - p^{i,j}).\)

Furthermore, we define the random time \(\tau^\pi\) as the first jump time of the Cox process \(N^\pi:\)

\[
\tau^\pi = \inf \{t \geq 0 : N^\pi_t > 0\},
\]

the default process \(D^\pi_t = 1_{\{\tau^\pi \leq t\}}\), and the filtration

\[
\mathcal{H}^\pi_t = \mathcal{F}^{D^\pi}_t = \sigma (D^\pi_s : 0 \leq s \leq t).
\]

For \(\pi \in \Pi_n\), the stopped process \(M^\pi_t \triangleq D^\pi_t - \int_0^{t \wedge \tau^\pi} \lambda^\pi (X_s) \, ds\) is a \((P, \{\mathcal{G}_t\})\)-martingale.
The Equivalent Fatal Shock Model (3)

Lemma 4 (Relationship between filtrations). We have

\[ \bigvee_{i=1}^{n} \mathcal{H}_t^i \subset \bigvee_{\pi \in \Pi_n} \mathcal{H}_t^\pi. \]

Lemma 5 (Obligor description using the fatal shock representation).

1. The Cox process \( N_t^i \) is given by

\[ N_t^i = \sum_{\pi \in \Pi_n} 1\{i \in \pi\} N_t^\pi, \]

and its intensity is

\[ \lambda^i (X_t) = \sum_{\pi \in \Pi_n} 1\{i \in \pi\} \lambda^\pi (X_t). \]

2. The default time \( \tau_i \) is given by

\[ \tau_i = \min \{\tau^\pi : \pi \in \Pi_n, i \in \pi\}. \]
The Marked Point Process Representation (1)

The Marshall-Olkin model is defined on the filtration \( \{ \tilde{G}_t \} \), which is larger than the one available to investors, namely \( \{ G_t \} \).

We shall use the generic tools of the MO model, however the local characteristics of the MPP representation are derived for the \( \{ G_t \} \) filtration.

We define the sequence of ordered default times \((T_0, T_1, ..., T_n) : T_0 = 0 \leq T_1 \leq ... \leq T_n\), and identities of the defaulted obligors as:

\[
\begin{align*}
T_0 &= 0, \ Z_0 = \emptyset; \\
T_k &= \min \{ \tau_i : 1 \leq i \leq n, \ \tau_i > T_{k-1} \} ; \\
Z_k &= \pi \text{ if } T_k = \tau_i \text{ for all } i \in \pi, \text{ and } \pi \in \Pi_n ;
\end{align*}
\]

The mark space of this point process is \( E \triangleq \Pi_n \), the set of all subsets of \( \{1,...,n\} \).
The Marked Point Process Representation (2)

The sequence \( (T_k, Z_k)_{k \geq 1} \) defines a MPP with couting measure

\[
\mu(\omega, dt \times dz) : (\Omega, \mathcal{G}) \rightarrow ((0, \infty) \times E, (0, \infty) \otimes \mathcal{E}),
\]

\[
\int_0^t \int_E H(\omega, t, z) \mu(\omega, dt \times dz) = \sum_{k \geq 1} H(\omega, T_k(\omega), Z_k(\omega)) \mathbf{1}_{\{T_k(\omega) \leq t\}},
\]

and \((P, \{\mathcal{G}_t\})\)-intensity kernel

\[
\lambda_t(\omega, dz) dt = \lambda_t(\omega) \Phi_t(\omega, dz) dt,
\]

where \( \lambda_t \) is the non-negative \( \{\mathcal{G}_t\} \)-predictable process

\[
\lambda_t = \sum_{\pi \in \Pi_n} \mathbb{E} \left[ 1 - D_t^{\pi} | \mathcal{G}_t \right] \lambda^{\pi}(X_t),
\]

and \( \Phi_t(\omega, dz) \) is the probability transition kernel from \((\Omega \times [0, \infty), \mathcal{G} \otimes B_+)\) into \((E, \mathcal{E})\)

\[
\Phi_t(\omega, \pi) = \frac{\mathbb{E} \left[ 1 - D_t^{\pi} | \mathcal{G}_t \right] \lambda^{\pi}(X_t)}{\lambda_t}, \text{ for } \pi \in \Pi_n,
\]

with \( \Phi_t(.) = 0 \) if \( \lambda_t = 0. \) \((\lambda_t, \Phi_t(dz))\) are the \((P, \{\mathcal{G}_t\})\)-local characteristics of \( \mu(dt \times dz) \).
The Marked Point Process Representation (3)

For $1 \leq i \leq n$, the compensated point process $M^i_t$ is given by

$$M^i_t = \int_0^t \int_E 1_{\{i \in z\}} (\mu (dt \times dz) - \lambda_t (dz) \, dt),$$

which can be written as

$$M^i_t = \sum_{\pi \in \Pi_n} 1_{\{i \in z\}} M^\pi_t |\{G_t\}$$

where $M^\pi_t |\{G_t\}$ is the $\{G_t\}$-adapted version of the compensated point process $M^\pi$:

$$M^\pi_t |\{G_t\} \triangleq \mathbb{E} [D_t^\pi |G_t] - \int_0^t \mathbb{E} [1 - D_s^\pi |G_s] \lambda^\pi (X_s) \, ds.$$ 

This MPP representation makes formal the idea that the mark space of the default times $(\tau_1, .., \tau_n)$ is $\Pi_n$ since joint defaults are allowed. Here, we have fixed the mark space, but as default events occur, we put zero probability mass for the states of $\Pi_n$, which cannot occur anymore.
Equivalent Local Martingale Measures (1)

Assumption. We assume that the dynamics of the zero-coupon defaultable bonds, under $P$, are given by

$$dS^i_t = S^i_t \left( \mu^i_t dt + (\sigma^i_t)^{tr} dW_t - dM^i_t \right),$$

where $\mu^i_t$ and $\sigma^i_t$ are $\{G_t\}$-predictable processes, uniformly bounded in $t$ and $\omega$, and regular enough to ensure that the prices $S^i_t$ are bounded for almost all $\omega \in \Omega$.

Assumption. We assume that the single name assets are not redundant.

No arbitrage. There are no arbitrage opportunities if and only if there exists a probability measure $Q$ equivalent to $P$ under which the (discounted) security prices $S$ are local martingales.

To classify the equivalent probability measures, we use the following Girsanov transformation (see Jacod and Shiryaev (1987)).
**Theorem 6** Let \( \theta \) be a \( d \)-dimensional \( \{G_t\} \)-predictable process and let \( \phi(t, z) \) be a \( \{G_t\} \)-predictable \( E \)-indexed nonnegative process such that:

\[
\int_0^t \|\theta_s\|^2 \, ds < \infty, \quad \int_0^t \int_E |\phi(s, z)| \lambda_s(dz) \, ds < \infty,
\]

for finite \( t \). Define the process \( L \) by

\[
dL_t = L_t - \left( \sum_{k=1}^d \theta_t^k dW_t^k + \int_E (\phi(s, z) - 1) (\mu(dt \times dz) - \lambda_t(dz) \, dt) \right),
\]

and \( L_0 = 1 \). Suppose that \( E^P[L_t] = 1 \), for all finite \( t \). Then, there exists a probability measure \( Q \) equivalent to \( P \) such that

1. we have \( dW_t = \theta_t dt + \tilde{W}_t \), where \( \tilde{W} \) is a \( d \)-dimensional Brownian motion under \( Q \);

2. the counting measure \( \mu(dt \times dz) \) has a \( (Q, \{G_t\}) \)-intensity kernel given by

\[
\tilde{\lambda}_t(dz) \, dt = \phi(t, z) \lambda_t(dz) \, dt.
\]
Equivalent Local Martingale Measures (3)

Every probability measure $Q$ locally equivalent to $P$ has the same structure as described above.

We restrict our set of ELMMs to the ones constructed with:

1. $\theta$ is a $d$-dimensional $\{\mathcal{F}_t\}$-predictable process;

2. the $E$-indexed $\{G_t\}$-predictable process $\phi(t, z)$ takes the form

   \[ \phi(t, \pi) = \phi^\pi(X_t), \text{ for } \pi \in \Pi_n, \]

   where $(\phi^\pi)_{\pi \in \Pi_n}$ is a set of strictly positive continuous bounded functions $\phi^\pi : \mathbb{R}^d \to \mathbb{R}_+$.

This allows us to preserve the Cox-process assumptions under the change of measure $Q$; in other words, the intensities are still driven by the $d$-dimensional Itô process $X$. 

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The dynamics of the zero-coupon defaultable bonds under the equivalent probability measure $Q$ are given by

$$dS^i_t = S^i_{t-} \left( \tilde{\mu}^i_t dt + (\sigma^i_t)^{tr} d\tilde{W}_t - \int_E \mathbf{1}_{\{i \in z\}} \left( \mu (dt \times dz) - \tilde{\lambda}_t (dz) dt \right) \right),$$

where

$$\tilde{\mu}^i_t = \mu^i_t + (\sigma^i_t)^{tr} \theta_t - \int_E \mathbf{1}_{\{i \in z\}} (\phi(t, z) - 1) \lambda_t (dz)$$

$$= \mu^i_t + (\sigma^i_t)^{tr} \theta_t - \sum_{\pi \in \Pi_n} \mathbf{1}_{\{i \in \pi\}} (\phi^\pi(X_t) - 1) \mathbb{E} [1 - D^\pi_t | G_t] \lambda^\pi(X_t).$$

To ensure absence of arbitrage, the drift $\tilde{\mu}^i_t$ under $Q$ is equal to zero:

$$0 = \mu^i_t + (\sigma^i_t)^{tr} \theta_t - \sum_{\pi \in \Pi_n} \mathbf{1}_{\{i \in \pi\}} (\phi^\pi(X_t) - 1) \mathbb{E} [1 - D^\pi_t | G_t] \lambda^\pi(X_t).$$

This system of linear equations classifies the set of ELMM that we consider. We have $n$ equations (one for each security) and $2^n - 1 + d$ unknowns (corresponding to each source of risk). The market is incomplete, hence the ELMM is not unique.
The Minimal Martingale Measure (1)

Assumption. We assume that the matrix

\[
\left(\sigma_t^i \right)^{\text{tr}} \sigma_t^j + \sum_{\pi \in \Pi_n} 1_{\{i \in \pi\}} 1_{\{j \in \pi\}} \mathbb{E} \left[ 1 - D_t^\pi \mid G_t \right] \lambda^\pi (X_t) \right]^{1 \leq i \leq n}_{1 \leq j \leq n}
\]

is invertible for all \(t \in [0, T^*]\).

Proposition 7 (Minimal martingale measure). Define the \(n\)-dimensional local martingale \(M^S\)

\[
\left( M_t^S \right)^i = \int_0^t S_s^i \left( \left( \sigma_s^i \right)^{\text{tr}} dW_s - \int_E 1_{\{i \in z\}} (\mu (ds \times dz) - \lambda_s (dz) ds) \right),
\]

and the \(n\)-dimensional predictable process \(\tilde{\lambda}\) as the solution of the linear system

\[
\mu_t^i = \sum_{j=1}^n \tilde{\lambda}_t^j S_t^j \left( \left( \sigma_t^i \right)^{\text{tr}} \sigma_t^j + \sum_{\pi \in \Pi_n} 1_{\{i \in \pi\}} 1_{\{j \in \pi\}} \mathbb{E} \left[ 1 - D_t^\pi \mid G_t \right] \lambda^\pi (X_t) \right].
\]

Then, the minimal martingale measure is given by the Doléans-Dade exponential \(\mathcal{E} \left( - \int \tilde{\lambda}^{\text{tr}} dM^S \right)_t\).
Proof. The Doob-Meyer decomposition of the price process \( S \) is given by

\[
S_t = S_0 + M_t^S + A_t^S,
\]

\[
(A_t^S)^i = \int_0^t S_{s-}^i \mu_s^i ds,
\]

\[
(M_t^S)^i = \int_0^t S_{s-}^i \left( (\sigma_s^i)^{tr} dW_s - \int_E \mathbf{1}_{\{i \in z\}} \left( \mu (ds \times dz) - \lambda_s (dz) ds \right) \right).
\]

The predictable covariance process of \( M \) is

\[
\left\langle M^{S_i}, M^{S_j} \right\rangle = \int_0^t S_{s-}^{i} S_{s-}^{j} \left( (\sigma_s^i)^{tr} \sigma_s^j + \int_E \mathbf{1}_{\{i \in z\}} \mathbf{1}_{\{j \in z\}} \lambda_s (dz) \right) ds
\]

The finite variation process \( A_t^S \) can be expressed as

\[
(A_t^S)^i = \left( \int_0^t d \left\langle M^{S_i} \right\rangle_s \hat{\lambda}_s \right)^i = \sum_{j=1}^n \int_0^t \hat{\lambda}_s^j d \left\langle M^{S_i}, M^{S_j} \right\rangle_s,
\]

where the predictable process \( \hat{\lambda} \) is given by inverting the following linear system

\[
\mu_t^i = \sum_{j=1}^n \hat{\lambda}_s^j S_t^j \left[ (\sigma_t^i)^{tr} \sigma_t^j + \sum_{\pi \in \Pi_n} \mathbf{1}_{\{i \in \pi\}} \mathbf{1}_{\{j \in \pi\}} \mathbb{E} \left[ 1 - D_t^\pi |G_t| \lambda^\pi (X_t) \right] \right].
\]
The uniform boundedness in $t$ and $\omega$ of the mean-variance trade-off process

$$\tilde{K}_t = \int_0^t (\hat{\lambda}_s)^{tr} \, d\langle M^S \rangle_s \hat{\lambda}_s = \sum_{i,j=1}^n \int_0^t \hat{\lambda}_s^i \hat{\lambda}_s^j \, d\langle M^{iS}, M^{jS} \rangle_s$$

ensures that the minimal martingale measure is given by the Doléans-Dade exponential as shown in Föllmer and Schweizer (1991). \qed
The Minimal Martingale Measure (2)

**Signed measure.** In general, the minimal martingale measure for discontinuous processes is only a signed measure since 
\[ E \left( - \int \hat{\lambda}^{tr} dM^S \right) \] can reach negative values. To ensure that \( \hat{P} \) is a true probability measure, we need additional assumptions. We can write the process \( (\hat{Z}_t)_{t \geq 0} \) as

\[
\hat{Z}_t = \exp \left( - \int_0^t (\hat{\lambda}_s)^{tr} d\overline{M}_s^S - \frac{1}{2} \int_0^t (\hat{\lambda}_s)^{tr} d\langle \overline{M}^S \rangle_s \hat{\lambda}_s \right) \prod_{s \leq t} \left( 1 - (\hat{\lambda}_s)^{tr} \Delta M_s^S \right),
\]

where \( \overline{M}_t^S \triangleq M_t^S - \sum_{s \leq t} \Delta M_s^S \) is the continuous part of the martingale \( M^S \).

The jump part is given by

\[
\sum_{s \leq t} \left( \Delta M_s^S \right)^i = - \int_0^t \int_E S_{s-1}^{i \in Z} \mu (ds \times dz) = - \sum_{T_k \leq t} S_{T_k}^{i \in Z_k},
\]

\[
\prod_{s \leq t} \left( 1 - (\hat{\lambda}_s)^{tr} \Delta M_s^S \right) = \prod_{T_k \leq t} \left( 1 + \sum_{i=1}^n \hat{\lambda}_{T_k}^{i} S_{T_k}^{i \in Z_k} \right).
\]
The Minimal Martingale Measure (3)

\( \hat{Z}_{T^*} \) is strictly positive if all the factors \( \left( 1 + \sum_{i=1}^{n} \hat{\lambda}_{T^*}^i S_{T^*}^i - 1 \{i \in Z_k\} \right) \) are positive.

A useful property of the minimal martingale measure is that if \((\alpha_t)_{t \geq 0}\) is a locally risk-minimizing strategy for the \(T\)-claim \(H_T\), then the value process is given by

\[
V_t(\alpha) = \hat{E}[H_T | \mathcal{G}_t].
\]

Next, we shall work under the minimal martingale measure and all expectations will be taken under this measure. Moreover, the FS decomposition will be done under \(\hat{P}\).

This is similar to the approach taken in Föllmer and Sondermann (1986) where a “good” martingale measure is chosen, then the local risk minimization is done with respect to this measure.
Martingale Representation (1)

The agents' filtration \( \{ G_t \} \) is generated by the Brownian motion \( \tilde{W} \) and the MPP \( \mu ( dt \times dz ) \) with \( ( \hat{P}, \{ G_t \} ) \)-intensity kernel \( \tilde{\lambda}_t ( dz ) \).

The martingale generator is \( ( \tilde{W}, ( \mu ( dt \times \{ z \} ) - \tilde{\lambda}_t ( \{ z \} ) )_{z \in \Pi_n} ) \) (see Jacod and Shiryaev (1987) Chap III Corollary 4.31).

**Proposition 8** (Martingale representation of \( H_t \)). The \( \{ G_t \} \)-martingale \( H_t = \hat{E} [ H_T | G_t ] \), \( t \in [0, T^*] \), where \( H_T \) is a \( G_T \)-measurable random variable, integrable with respect to \( \hat{P} \), admits the following integral representation

\[
H_t = H_0 + \int_0^t (\xi_s)^{tr} \, d\tilde{W}_s - \int_0^t \int_E \zeta (s, z) \, (\mu ( dt \times dz ) - \tilde{\lambda}_s ( dz ) \, ds),
\]

where \( \xi \) is a \( d \)-dimensional \( \{ G_t \} \)-predictable process and \( \zeta (s, z) \) is an \( E \)-indexed \( \{ G_t \} \)-predictable process \( \zeta (s, z) \) such that

\[
\int_0^t \| \xi_s \|^2 \, ds < \infty, \quad \int_0^t \int_E \zeta (s, z) \, \tilde{\lambda}_s ( dz ) \, ds < \infty,
\]

almost surely.
Martingale Representation (2)

This can be written as

\[ H_t = H_0 + \int_0^t (\xi_s)^{tr} \, d\tilde{W}_s - \sum_{\pi \in \Pi_n} \int_{0}^{t} \zeta(s, \pi) \, d\tilde{M}_s^{\pi}|_{\{G_t\}}, \]

where \( \tilde{M}^{\pi}|_{\{G_t\}} \) is the \( \{G_t\} \)-adapted version of the compensated point process \( \tilde{M}^{\pi} \):

\[ \tilde{M}^{\pi}|_{\{G_t\}} \triangleq \mathbb{E} [D_t^{\pi} | G_t] - \int_0^t \mathbb{E} [1 - D_s^{\pi} | G_s] \tilde{\lambda}^{\pi} (X_s) \, ds. \]

In order to replicate the claim \( H_T \), one needs to match the diffusion terms \( \xi_{s}^{i}, \ 1 \leq i \leq d \), and the jump-to-default terms \( [-\zeta(s, \pi)] \) for each possible default state.
Computing the Hedging Strategy: Main Result (1)

We use the martingale representation in Proposition 8 to derive the local risk-minimization hedging strategy. As shown in Föllmer and Schweizer (1991), this is equivalent to finding the FS-decomposition

\[ H_T = H_0 + \int_{0}^{T} (\alpha_t)^{tr} dS_t + L_t. \]

Our goal is to establish an analytical result, which derives single-name hedges \( (\alpha^i)_{1 \leq i \leq n} \) in terms of the martingale representation processes \( \xi \) and \( \zeta (., \pi), \pi \in \Pi_n \).
The strategy \((\alpha_t)_{t \geq 0}\) can be computed as

\[
\alpha_t = d \langle M^S \rangle_t^{-1} d \langle M^S, V(\alpha) \rangle_t,
\]

where the value process is given by

\[
V_t(\alpha) = \hat{\mathbb{E}}[H_T | G_t], \text{ for } t \in [0, T].
\]

This follows from the FS-decomposition of \(H\) under \(\hat{P}\) and the projection of \(V_t(\alpha)\) on the \(\hat{P}\)-martingale \(\int_{0}^{t} (\alpha_s)^{tr} dS_s\).

**Theorem 9** *(Local risk-minimization hedging strategy)*. The local risk-minimization hedging strategy of a general (basket) contingent claim with single name instruments is given by the solution of the following linear system,

for \(1 \leq k \leq n\),

\[
\sum_{i=1}^{n} \alpha_t^i \left[ S_{t^-}^i S_{t^-}^k \left[ \left( \sigma_t^i \right)^{tr} \sigma_t^k + \int_E 1_{i \in z} 1_{k \in z} \tilde{\lambda}_t (dz) \right] \right] = S_{t^-}^k \left( \sigma_t^k \right)^{tr} \xi_t + \int_E \zeta(t, z) S_{t^-}^k 1_{k \in z} \tilde{\lambda}_t (dz).
\]
Proof. $M^S$ is defined as
\[
(M^S_t)^i = \int_0^t S^i_s\left( (\sigma^i_s)^{tr} d\tilde{W}_s - \int_E 1\{i \in z\} (\mu (ds \times dz) - \tilde{\lambda}_s (dz) ds) \right),
\]
and the predictable covariance is
\[
d\langle M^S \rangle_{i,j}^t = d\langle M^S_i, M^S_j \rangle^t_t = S^i_t S^j_t - \int_0^t \int_E \zeta (s, z) \left( \mu (ds \times dz) - \tilde{\lambda}_s (dz) ds \right) dt.
\]
The value process $V_t (\alpha) = \hat{\mathbb{E}} [H_T | G_t]$ is given by the martingale representation
\[
V_t (\alpha) = \hat{\mathbb{E}} [H_T | G_t] = H_0 + \int_0^t (\xi_s)^{tr} d\tilde{W}_s - \int_0^t \int_E \zeta (s, z) \left( \mu (ds \times dz) - \tilde{\lambda}_s (dz) ds \right).
\]
Hence, we have
\[
d\langle M^S, V (\alpha) \rangle^t = S^i_t \left( (\sigma^i_t)^{tr} \xi_t + \int_E 1\{i \in z\} \zeta (t, z) \tilde{\lambda}_t (dz) \right) dt.
\]
Computing the Hedging Strategy: Main Result (3)

This can be written explicitly in terms of the model parameters.

Applying Itô’s lemma and using the Markovian property of \( X \), we find an explicit expression of the dynamics of \( S^i \) under the martingale measure \( \hat{P} \).

**Lemma 10 (Single-name price process representation).** We have

\[
S^i_t = S^i_0 - \int_0^t \int_E \tilde{s}^i(s, X_s) 1_{\{i \in z\}} \left( \mu(ds \times dz) - \tilde{\lambda}_s (dz) ds \right)
+ \int_0^t \left( 1 - D^i_s \right) \sum_{j=1}^d \sum_{k=1}^d \frac{\partial \tilde{s}^i(s, X_s)}{\partial x_j} \beta_{jk}(X_s) d\tilde{W}^k_s,
\]

where \( \tilde{s}^i(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is defined as

\[
\tilde{s}^i(t, x) \triangleq \mathbb{E}_{(t, x)} \left[ \exp \left( - \int_t^T \tilde{\lambda}^i(X_s) ds \right) \right].
\]
Computing the Hedging Strategy: Main Result (4)

Lemma 10 establishes the martingale representation for the single-name securities whose payoff is $H_T = 1 - D^i_T$:

$$
\zeta^{1-D^i_T}(t, z) = 1_{\{i \in z\}} \bar{s}^i(t, X_t), \text{ for } z \in \Pi_n,
$$

$$
\left(\xi_t^{1-D^i_T}\right)^k = (1 - D^i_t) \sum_{j=1}^d \frac{\partial \bar{s}^i(t, X_t)}{\partial x_j} \beta_{jk}(X_t).
$$

The hedging strategy is solution of

for $1 \leq k \leq n$

$$
\sum_{i=1}^n \alpha^i_t \left[ \int_E \zeta^{1-D^i_T}(t, z) \zeta^{1-D^i_T}(t, z) \tilde{\lambda}_t(dz) + \left(\xi_t^{1-D^i_T}\right)^{tr} \xi_t^{1-D^i_T} \right]
$$

$$
= \int_E \zeta(t, z) \zeta^{1-D^i_T}(t, z) \tilde{\lambda}_t(dz) + \left(\xi_t^{1-D^i_T}\right)^{tr} \xi_t.
$$

Note that this problem combines both default risk and spread risk.
Applications (1)

We consider a first-to-default (basket) contingent claim whose payoff is

$$H_T^{(1)} = \prod_{i=1}^{n} (1 - D_T^i).$$

The price of this claim, at time $t$, is

$$H_t^{(1)} = \hat{E} \left[ \prod_{i=1}^{n} (1 - D_T^i) \mid G_t \right].$$

We can show that it can be expressed as

$$H_t^{(1)} = \left[ \prod_{i=1}^{n} (1 - D_t^i) \right] \tilde{h}^{(1)} (t, X_t),$$

where the function $\tilde{h}^{(1)} (t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\tilde{h}^{(1)} (t, X_t) = \hat{E}_{(t,x)} \left[ \exp \left( - \int_t^T \tilde{\lambda}^{(1)} (s, X_s) ds \right) \right],$$

$$\tilde{\lambda}^{(1)} (t, x) = \sum_{j=1}^{m} \left[ 1 - \prod_{i=1}^{n} (1 - \tilde{p}^{i,j}) \right] \tilde{\lambda}^{c_j} (X_t).$$
Applications (2)

Using Itô’s lemma and some algebra, we find

\[
dH_t^{(1)} = -\int_E \tilde{h}^{(1)}(t, X_t) \left( \mu \left( dt \times dz \right) - \tilde{\lambda}_t \left( dz \right) dt \right) + \left[ \prod_{i=1}^n \left( 1 - D_i^t \right) \right] \sum_{j=1}^d \sum_{k=1}^d \frac{\partial \tilde{h}^{(1)}(t, X_t)}{\partial x_j} \beta_{jk}(X_t) d\tilde{W}_k^t.
\]

This gives the processes of the martingale representation

\[
\zeta_{H_t^{(1)}}(t, z) = \tilde{h}^{(1)}(t, X_t), \text{ for all } z \in \Pi_n,
\]

\[
\left( \xi_{t H_t^{(1)}} \right)_k = \left[ \prod_{i=1}^n \left( 1 - D_i^t \right) \right] \sum_{j=1}^d \frac{\partial \tilde{h}^{(1)}(t, X_t)}{\partial x_j} \beta_{jk}(X_t),
\]

which can be plugged into the linear system of Theorem 9. Inverting the latter gives the single-name hedge ratios of the first-to-default basket claim.
Conclusion

- We have addressed the problem of hedging basket credit derivatives with single-name instruments in a Marshall-Olkin Model.

- We have used the Equivalent Fatal Shock model to derive the Marked Point Process representation.

- This is then used to classify the set of ELMMs and to derive the Minimal Martingale Measure $\hat{P}$.

- Working under $\hat{P}$, we have applied a Martingale Representation Theorem to the value process, and established the local risk-minimization hedging strategy.

- We have worked out the First-To-Default example to illustrate our result.


