The homotopy determinantal torsor

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Outline. The Beilinson-Kato category

- Goal: Define and describe in the language of homotopy theory the notion of **(higher) homotopy determinantal theory** for a large class of spaces, and introduce a topological space which collects together all such determinantal theories.

  Let $\mathcal{A}$ be an exact category. Associated to it, there is the **Beilinson-Kato category** of "locally compact objects" on $\mathcal{A}$, and denoted by $\underset{\longleftrightarrow}{\lim}\mathcal{A}$, which inherits an exact structure from $\mathcal{A}$. The category $\underset{\longleftrightarrow}{\lim}\mathcal{A}$ is a full subcategory of the iterated ind/pro category $\operatorname{Ind}\operatorname{Pro}(\mathcal{A})$.
- Examples:
  - If $\mathcal{A}$ is the category $\mathbf{Vect}_0(k)$ of finite dimensional vector spaces on a field k, $\underset{\longleftrightarrow}{\lim}\mathcal{A}$ is the category of Tate spaces, i.e., **locally linearly compact topoolgical vector spaces**. For example, $k((t))$ and $k[[t]]$.
  - More in general, we can iteratively define **higher Tate spaces** of dimension n as the objects of the iterated category $T_n : = \underset{\longleftrightarrow}{\lim}\mathcal{T}_{n-1}$, where $\mathcal{T}_0 = \mathbf{Vect}_0(k)$. For example, the space of iterated power series $\mathcal{C}((t_1))\cdots((t_n))$ is an object of $\mathcal{T}_n$.
  - Higher local fields of dimension n are objects of the iterated category $\underset{\longleftrightarrow}{\lim}^n\mathcal{A}$, where $\mathcal{A}$ is the category of finite abelian groups.

  The passage $\mathcal{A} \to \underset{\longleftrightarrow}{\lim}\mathcal{A}$ can be described as a sort of "construction of a category of infinite-dimensional objects over the objects of $\mathcal{A}$", where the latter are interpreted as finite-dimensional ones.

The Sato Grassmannian

Let X be an object of the category $\underset{\longleftrightarrow}{\lim}\mathcal{A}$. The **Sato Grassmannian** for X, which generalizes the well-known concept defined by Sato for Tate spaces, is defined as the family $\Gamma(X) = \{U\}$ of subobjects of X in the category $\operatorname{Pro}(\mathcal{A})$, for which the quotient X/U is in $\operatorname{Ind}(\mathcal{A})$. In particular, for two objects $U_1 \hookrightarrow U_2$ of $\Gamma(X)$, we obtain that their quotient U_1/U_2 is in $\mathcal{A}$ (see [5]).

The Waldhausen space

Given an exact category, Waldhausen [6] associates to it a simplicial category $S_\bullet(\mathcal{A})$, whose geometric realization $S(\mathcal{A})$ provides a topological model for the K-theory of $\mathcal{A}$, i.e., $K_i(\mathcal{A}) = \pi_{i+1}S(\mathcal{A})$.
Let $\mathcal{A}$ be an exact category and $n \geq 0$ an integer. The category $S_n(\mathcal{A})$ is defined (roughly speaking) as the category whose objects are sequence $\underline{a} = 0 \hookrightarrow a_1 \hookrightarrow a_2 \hookrightarrow \cdots \hookrightarrow a_n$ of n admissible monomorphisms, toghether with a compatible choice of an object a_{ij}, in the isomorphism class of each quotient a_j/a_i

For each $n \geq 0$, we have a functor $\partial_0 : S_n(\mathcal{A}) \to S_{n-1}(\mathcal{A})$ by $\partial_0(\underline{a}) = a_2/a_1 \hookrightarrow \cdots \hookrightarrow a_n/a_1$, and for all $i > 0$ a functor $\partial_i : S_n(\mathcal{A}) \to S_{n-1}(\mathcal{A})$ for all $0 < i \leq n$ by $\partial(a_1 \hookrightarrow a_2 \hookrightarrow \cdots \hookrightarrow a_n) = a_1 \hookrightarrow \cdots \hat{a}_i \hookrightarrow \cdots \hookrightarrow a_n$. Also, there are functors $s_i : S_n(\mathcal{A}) \to S_{n+1}(\mathcal{A})$, for $0 \leq i \leq n$ which are defined by doubling the object a_i in $\underline{a}$. Then the system $(S_n(\mathcal{A}), \partial_i, s_j)$ forms a simplicial category $S_\bullet(\mathcal{A})$.

The **Waldhausen space** $S(\mathcal{A})$ is the geometric realization of the bisimplicial set $[n] \mapsto Nerv(S_n(\mathcal{A}))$. Thus it is a CW complex, glued out of bisimplices $\Delta^p \times \Delta^q$; the (p, q)-cells are labelled by chains of q composable isomorphisms of admissible filtrations of lenght p. This space is homotopy equivalent to the space $s(\mathcal{A})$, the geometric realization of the "horizontal" simplicial set $[n] \mapsto Ob(S_n(\mathcal{A}))$ of the objects of the categories $S_n(\mathcal{A})$.

Dimension and Determinant theories for Tate spaces

Kapranov ([3]) has introduced the notion of **dimension theory** and of **determinantal theory** for a Tate space V. This is a map $d : \Gamma(V) \to \mathbb{Z}$ such that, whenever $U_1, U_2 \in \Gamma(V)$ and $U_1 \subset U_2$, we have

$$d(U_2) = d(U_1) + \dim(U_2/U_1). \tag{1}$$

The set of dimension theories on V is denoted by $\operatorname{Dim}(V)$. The group $\mathbb{Z}$ acts on $\operatorname{Dim}(V)$ by adding constant functions. This makes $\operatorname{Dim}(V)$ into a **$\mathbb{Z}$-torsor.** This structure is the first in a hierarchy of structures which take into account the invariants that can be associated to a Tate space V. At the next level, we have the notion of **determinantal theory** on V. This is defined as follows. For $W \in \mathbf{Vect}_0(k)$, let $\det(W)$ be the top exterior power of W. Recall that for any short exact sequence $0 \to W' \to W \to W'' \to 0$ in $\mathbf{Vect}_0(k)$, we have a natural identification $m : \det(W') \otimes \det(W'') \to \det(W)$, and these identification are associative for any filtration $W_1 \subset W_2 \subset W$ of lenght 2. A **determinantal theory** on V is defined as a pair (Δ, δ), where Δ is a map that associates to each $U \in \Gamma(V)$ a 1-dimensional vector space $\Delta(U)$, such that for each embedding $U_1 \subset U_2$ there is an isomorphism

$$\delta : \Delta(U_1) \otimes \det(U_2/U_1) \to \Delta(U_2)$$

so that for each nested triple $U_1 \subset U_2 \subset U_3$ the diagram

$$\begin{tikzcd}
\Delta(U_1) \otimes \det\left(\frac{U_2}{U_1}\right) \otimes \det\left(\frac{U_3}{U_2}\right) \arrow[r, "1\otimes\lambda"] \arrow[d, "\delta_{U_1,U_2}\otimes 1"'] & \Delta(U_1) \otimes \det\left(\frac{U_3}{U_1}\right) \arrow[d, "\delta_{U_1,U_3}"] \\
\Delta(U_2) \otimes \det\left(\frac{U_3}{U_2}\right) \arrow[r, "\delta_{U_2,U_3}"'] & \Delta(U_3)
\end{tikzcd}, \tag{2}$$

is commutative. We denote by $\operatorname{Det}(V)$ the category (groupoid) formes by all determinantal theories and their isomorphisms (defined in the obvious way). If $\phi : \Delta \to \Delta'$ is an isomorphism of determinantal theories and $\lambda \in k^*$, then λk is also an isomorphism. This defines an action of k^* on the morphisms which makes $\operatorname{Det}(V)$ into a k^*-gerbe, i.e., each $\operatorname{Hom}(\Delta, \Delta')$ becomes a k^*-torsor and the composition is bilinear. The gerbe $\operatorname{Det}(V)$ is called the **gerbe of determinantal theories** on V.

Determinantal torsor: the discrete (Picard) theory

The relation that the functions $\dim$ and $\det$ satisfy, and the corresponding notions of Dim and Det are better understood in terms of the Waldhausen space. Precisely, consider any exact category $\mathcal{A}$ and let X be an object of the Beilinson-Kato category $\underset{\longleftrightarrow}{\lim}\mathcal{A}$, interpreted as a "generalized" Tate space. We can reinterpret the above constructions in a single process, using the notion of a **torsor over a Picard category**. We start with an abstract determinantal theory on $\mathcal{A}$, defined as a datum (h, λ) defined on the cells of dimensions less than or equal to 3 of the Waldhausen space of $\mathcal{A}$, taking values in the Picard category $V(\mathcal{A})$ of the virtual objects of $\mathcal{A}$ (this is a category such that $\pi_0(\mathcal{A}) = K_0(\mathcal{A})$ and $\pi_1(\mathcal{A}) = K_1(\mathcal{A})$). It results that such a determinantal theory can be interpreted as a 2-cocycle on $S(\mathcal{A})$, and it can be used to define a torsor $\operatorname{Det}_h(X)$ over the Picard category $V(\mathcal{A})$. When we restrict to $\pi_0(V(\mathcal{A}))$, we see that this torsor reduces to the Dimensional torsor $\operatorname{Dim}(X)$ defined over the group $K_0(\mathcal{A})$, and when we restrict to $\pi_1 V((\mathcal{A}))$, we re-obtain the Determinant torsor $\operatorname{Det}(X)$. In short, this new concept of "Determinant torsor over a Picard category" includes both structures (torsor over the group $K_0(\mathcal{A})$; gerbe over the group $K_1(\mathcal{A})$).

The homotopy generalization

The introduction of the Picard torsor $\operatorname{Det}(X)$ allows us to see the two structures Dim and Det from a unified perspective. However, it is limited in some sense because

- It contains only dimensional theories ("dimension one") and determinantal theories ("dimension two") for the space $X \in \underset{\longleftrightarrow}{\lim}\mathcal{A}$;
- It detects only the action of the *first two* K-groups $\mathcal{K}_0(\mathcal{A})$ and $\mathcal{K}_1(\mathcal{A})$ on it, but says nothing about the other $\mathcal{K}$-groups of $\mathcal{A}$;
- It is a "discrete" object, and it has no counterpart in homotopy-theoretic terms.

Thus, we would like to have at our disposal a more powerful construction: a topological space $\mathcal{D}(X)$ which extends the above construction of $\operatorname{Det}(X)$ to a (to be defined) *torsor over the whole $\mathcal{K}$-theoretical space of the exact category $\mathcal{A}$.* Namely, a "torsor" over the loop space $\mathcal{K}(\mathcal{A}) = \Omega S(\mathcal{A})$, whose points can be interpreted as the collections of all the determinantal theories of higher orders to be defined on the space X and which extends to the full homotopy-theoretic level the Picard torsor $\operatorname{Det}(X)$. The latter should appear as the "category-level approximation" (i.e., depending only on $\mathcal{K}_0(\mathcal{A})$ and $\mathcal{K}_1(\mathcal{A})$) of this newly defined "∞-homotopy torsor" $\mathcal{D}(X)$, which is going now to include the action of the whole $\mathcal{K}$-theoretical space of the category $\mathcal{A}$.

The homotopy determinantal torsor $\mathcal{D}(X)$

We have been motivated by a remark of D. Drinfeld, who in [2] writes "*[t]he notion of determinant Torsor is very useful and its rigorous interpretation in the standard homotopy-theoretic language of algebraic K-theory could be helpful.*". In order to accomplish this, we raise the discussion above to the simplicial-homotopy and the operadic levels.
Recall the definition of the Grassmannian $\Gamma(X)$ of X. We define the **(semi-infinite) building $B_\bullet(X)$** as the nerve of the poset $\Gamma(X)$, ordered by inclusion. In particular, we obtain a simplicial set whose n-simplices correspond to admissible flags of length $n+1$: $L_0 \hookrightarrow \ldots \hookrightarrow L_n$ of subobjects of X. For each such flag, we obtain a filtration

$$\frac{L_1}{L_0} \hookrightarrow \ldots \hookrightarrow \frac{L_n}{L_0}$$

of objects of $\mathcal{A}$, hence an object of the n-th Waldhausen category $S_n(\mathcal{A})$. This map extends to a (bi-)simplicial map

$$h_\bullet : B_\bullet(X) \longrightarrow S_\bullet(\mathcal{A}).$$

Next, pass to the geometric realization and construct the mapping cone $C(h)$. Finally, let $\mathcal{D}(X)$ be the space of all continuous based maps $\tau : C(h) \to S(\mathcal{A})$ such that $\tau \cdot i = Id$. We have the following result, which depends on the fact that the space $B(X)$ is contractible.

Proposition 3. *There is a weak homotopy equivalence $\mathcal{D}(X) \simeq \mathcal{K}(\mathcal{A})$.*

Equivalently, the space $\mathcal{D}(X)$ can be defined in terms of the following extension problem. Construct the simplicial cone $CB_\bullet(X)$ on the building $B_\bullet(X)$. Then, define $\mathcal{D}(X)$ to be the space of all continuous based maps $\Delta : CB(X) \longrightarrow S(\mathcal{A})$, whose restriction on $B(X)$ is equal to the map h. Namely, the space of based maps δ which make the following diagram of the geometric realizations

$$\begin{tikzcd}
B(X) \arrow[d, hook, "i"] \arrow[dr, "h"] & \\
CB(X) \arrow[r, "\delta"'] & S(\mathcal{A})
\end{tikzcd}$$

commutative. In plain words, every point $\delta_* \in \mathcal{D}(X)$ can be interpreted as a collection of functions $\delta_* = \{\delta_0, \delta_1, \cdots, \delta_n, \cdots\}$ where δ_n is defined on the n-cells of the building $B(X)$ and which satisfy the following conditions, induced from the simpliciality of the above construction:

(0) For any $L_0 \in \Gamma(X)$, a loop $\delta_1(L_0)$ in the Waldhausen space $S(\mathcal{A})$, at the basepoint $*$ (i.e., an element of $\mathcal{K}(A) = \Omega(S(\mathcal{A}))$.

(1) For any $L_0 \hookrightarrow L_1$ in $\Gamma(X)$, a map of the 2-simplex $\delta_2(L_0 \hookrightarrow L_1) : \mathbf{\Delta^2} \to \mathbf{S}(\mathcal{A})$ on the Waldhausen space $S(\mathcal{A})$, which can be represented as a homotopy rel$*$ between the composition of the loops $\delta_0(L_1) * \frac{L_1}{L_0}$ and the loop δ_{L_1}. We can write it down explicitly this relation as

$$\delta_2(L_0, L_1) : \delta(L_0) * h\left(\frac{L_1}{L_0}\right) \simeq \delta(L_1)$$

which enlightens δ_2 as the "homotopy version" of a dimensional theory as defined in equation (1).

(2) For any $L_0 \hookrightarrow L_1 \hookrightarrow L_2$ in $\Gamma(X)$, a map of the 3-simplex $\delta_3(L_0 \hookrightarrow L_1 \hookrightarrow L_2) : \mathbf{\Delta^3} \to \mathbf{S}(\mathcal{A})$ in the Waldhausen space which can be represented as a homotopy of homotopies rel$*$ between the following composition of homotopies rel$*$, explicitly written as a diagram, commutative up to a homotopy δ_3:

$$\begin{tikzcd}
\delta(L_0) * h\left(\frac{L_1}{L_0}\right) * h\left(\frac{L_2}{L_1}\right) \arrow[r, "1*h"] \arrow[d, "\delta_{L_0,L_1}*1"'] \arrow[dr, phantom, "\Downarrow\delta_3"] & \delta(L_0) * h\left(\frac{L_2}{L_0}\right) \arrow[d, "\delta_{L_0,L_2}"] \\
\delta(L_1) * h\left(\frac{L_2}{L_1}\right) \arrow[r, "\delta_{L_1,L_2}"'] & \delta(L_2)
\end{tikzcd}, \tag{4}$$

(4) and so on.

Form an informal point of view, the points of $\mathcal{D}(X)$ represent attempts to associate, in a consistent way, $\mathcal{K}$-theoretic invariants (lying in the $\mathcal{K}$-theory of $\mathcal{A}$) to "semi-infinite objects" $L \in \Gamma(X)$ of the space $X \in \underset{\longleftrightarrow}{\lim}\mathcal{A}$, which by themselves do not lie in $\mathcal{A}$ at all: only the quotients L_i/L_j are in $\mathcal{A}$. Given a point $\{\delta_n\}$ of $\mathcal{D}(X)$, we can thus naturally interpret each δ_n as a **n-homotopy determinantal theory**, whose behaviour on the boundaries of the n-simplex $\mathbf{\Delta^n}$ is dictated by the (previously defined) $(n-1)$-homotopy δ_{n-1}.

$\mathcal{D}(X)$ as a homotopy torsor. The operadic structure.

The space $\mathcal{D}(X)$ is thus the "full homotopy version" of the determinantal Picard torsor $\operatorname{Det}(X)$. As we have seen, the torsoriality of $\operatorname{Det}(X)$ over the Picard category of virtual objects of $\mathcal{A}$ encodes the first two $\mathcal{K}$-invariants of the category $\mathcal{A}$. Now we want to express the torsoriality of the newly defined $\mathcal{D}(X)$ over the whole $\mathcal{K}$-theoretical space of $\mathcal{A}$. The first step is to introduce a convenient notion of **torsor over an (infinite) loop space**, such as $\mathcal{K}(\mathcal{A}) = \Omega S(\mathcal{A})$. The principle of the right definition lies in the notion of **module over an algebra over an operad**, introduced by Kriz and May in [4] and developed, among others, by C. Berger and I. Moerdijk in [1]. Explicitely, let $\mathcal{C}$ be the operad of 1-little cubes, $A = \mathcal{K}(\mathcal{A}) = \Omega S(\mathcal{A})$ considered as an algebra over $\mathcal{C}$ and T a module over the algebra A (hence, over the $\mathcal{K}$-theoretical space of $\mathcal{A}$, understood as a loop space $\Omega S(\mathcal{A})$). Let $\psi_j : \mathcal{C}(j) \times A^{j-1} \times T \to T$ be the action of the $\mathcal{C}$-algebra A on the module M. We say that T is a **homotopy torsor** if, for $j = 2$, and for all $\omega \in \mathcal{C}_2$ and all $x \in T$, ψ_2 induces a (weak) homotopy equivalence $A \to M$, given by $a \mapsto \psi_2(\omega, a, x)$.
There is a natural action of $\Omega S(\mathcal{A})$ onto the space $\mathcal{D}(X)$, which can be examplified by the picture, when we take the particular case $j = 3$ and δ_2 for a given point $\delta_* \in \mathcal{D}(X)$:

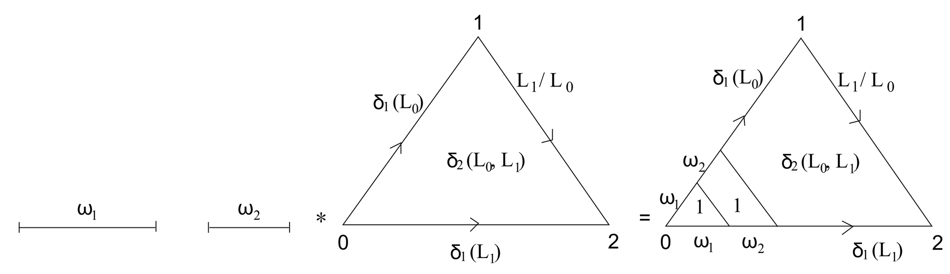

Figure 1: Action of $\Omega(S(\mathcal{A}))$ on δ_2

Now, recall Proposition (3). This fact coupled with the existence of the above action gives the proof of the following

Theorem 5. *The space $\mathcal{D}(X)$ of higher homotopy determinantal theories is a homotopy torsor.*

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