A classical Perron method for existence of smooth solutions to boundary value and obstacle problems for degenerate-elliptic operators via holomorphic maps

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A class of boundary-degenerate, linear, second-order elliptic and parabolic operators
A class of boundary-degenerate operators I

Throughout our presentation, we take

\[ Av := -x_d \text{tr}(aD^2v) - \langle b, Dv \rangle + cv \quad \text{on } \partial\mathcal{O}, \quad v \in C^\infty(\partial\mathcal{O}), \quad (1) \]

where \( x = (x_1, \ldots, x_d) \) are the standard coordinates on \( \mathbb{R}^d \), and \( \mathcal{O} \subseteq \mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}^+ \) is a subdomain of the half-space, and

\[ \partial_1 \mathcal{O} := \partial\mathcal{O} \cap \{ x_d > 0 \} \quad \text{and} \quad \partial_0 \mathcal{O} := \text{Interior}(\partial\mathcal{O} \cap \{ x_d = 0 \}). \]

The coefficient functions \( a, b, c \) are defined on \( \bar{\mathcal{O}} \), the matrix \( (a_{ij}) \) is symmetric, and there are positive constants \( b_0, \lambda_0, \Lambda \) such that

\[ \| a \|_{C^\alpha_s(\bar{\mathcal{O}})} + \| b \|_{C^\alpha_s(\bar{\mathcal{O}})} + \| c \|_{C^\alpha_s(\bar{\mathcal{O}})} \leq \Lambda, \quad (2a) \]

\[ \langle a\xi, \xi \rangle \geq \lambda_0|\xi|^2 \quad \text{on } \bar{\mathcal{O}}, \quad \forall \xi \in \mathbb{R}^d, \quad (2b) \]

\[ b^d \geq b_0 \quad \text{on } \partial_0 \mathcal{O}. \quad (2c) \]
When discussing uniqueness of solutions, we shall also require that $c \geq 0$ on $\partial$ (and occasionally $c > 0$ on $\partial$ or even $c \geq c_0$ on $\partial$, where $c_0$ is a positive constant).

For example, the generator, $A$, of the Heston stochastic volatility process, which we shall describe later, has this style of degeneracy, as do certain linear operators arising in the study of the porous medium equation.

When discussing parabolic problems, we shall suppose that

$$Lv := -v_t + Av \quad \text{on } \Omega_T, \quad v \in C^\infty(\Omega_T), \quad (3)$$

where $T > 0$ and $\Omega_T := (0, T) \times \partial$ and $A$ is as in (1), except that its coefficients are now also allowed to depend on $t$ and the properties (2) are modified in an obvious way to this $t$-dependency into account.
The Hölder space $C_s^\alpha(\bar{O})$ is a modification of the usual one, $C^\alpha(\bar{O})$, and is defined by replacing the Euclidean distance function, $ds^2 = \sum_{i=1}^{d} dx_i^2$, by the ‘cycloidal’ distance function, $ds^2 = x_d^{-1} \sum_{i=1}^{d} dx_i^2$.

We shall later give more precise definitions of $C_s^\alpha(\bar{O})$ and $C_s^{2+\alpha}(\bar{O})$, due to P. Daskalopoulos and R. Hamilton (1998), along with certain variants.

For now, we shall just note that they allow one to

- Take the boundary-degeneracy of the operators $A$ and $L$ into account for a priori Schauder estimates,
- Avoid prescribing any boundary condition for a solution, $u$, along $\{x_d = 0\}$, aside from smoothness up to $\{x_d = 0\}$, in boundary-value or obstacle problems defined by $A$ or $L$. 


Though not an explicit part of the definition, if $u \in C_s^{2+\alpha}(\bar{\Omega})$ then

$$x_d D^2 u = 0 \quad \text{on } \{x_d = 0\},$$

and thus, when $Au = f$ on $\partial \Omega$ and $f \in C_s^\alpha(\bar{\Omega})$,

$$-\langle b, Du \rangle + cu = f \quad \text{on } \{x_d = 0\},$$

and similarly for $u \in C_s^{2+\alpha}(\bar{\Omega}_T)$ and $Lu = f$ on $\partial \Omega_T$. 
Figure: Boundary conditions are prescribed along the ‘non-degenerate’ boundary portion, \( \partial_1 \Omega \), but no boundary conditions need be prescribed along the ‘degenerate’ boundary portion, \( \partial_0 \Omega \).
Motivation from mathematical finance
Motivation from mathematical finance I

We wish to prove uniqueness, existence, and regularity of solutions to the following four problems:

1. Elliptic boundary value problem with partial Dirichlet data,

   \[ Au = f \quad \text{on } \mathcal{O}, \quad u = g \quad \text{on } \partial_1 \mathcal{O}. \]

2. Parabolic terminal-boundary value problem with partial Dirichlet data (for a European-style barrier option),

   \[ Lu = f \quad \text{on } \mathcal{O}_T, \quad u = g \quad \text{on } ((0, T) \times \partial_1 \mathcal{O}) \cup \{ T \} \times (\mathcal{O} \cup \partial_1 \mathcal{O}). \]

3. Elliptic obstacle problem with partial Dirichlet data (for a perpetual American-style barrier option),

   \[ \min\{ Au - f, u - \psi \} = 0 \quad \text{on } \mathcal{O}, \quad u = g \quad \text{on } \partial_1 \mathcal{O}, \]

   provided \( \psi \leq g \) on \( \partial_1 \mathcal{O} \).
Motivation from mathematical finance II

4. Parabolic terminal-boundary value problem with partial Dirichlet data (for a finite-horizon American-style barrier option),

\[
\min \{ Lu - f, u - \psi \} = 0 \quad \text{on } \partial_T,
\]

\[
u = g \quad \text{on } ((0, T) \times \partial_1 \Omega) \cup (\{ T \} \times (\Omega \cup \partial_1 \Omega)),
\]

provided \( \psi \leq g \) on \((0, T) \times \partial_1 \Omega \) \( \cup \) \((\{ T \} \times (\Omega \cup \partial_1 \Omega))\).

For the American-style put option with strike \( E > 0 \),

\[
\psi(x_1, x_2) = (E - e^{x_1})^+, \quad (x_1, x_2) \in \mathbb{H}.
\]

For brevity, we mostly restrict our attention to the elliptic problems in this presentation.
Motivation from mathematical finance III
Example (Asset price modeled by a Heston stochastic volatility process)

If $-A$ is the generator with killing of the Heston stochastic volatility process (in coordinates $(x_1, x_2)$ representing the asset log price and stochastic variance, respectively), then

$$Av := -\frac{x_2}{2} \left( v_{x_1 x_1} + 2\rho \sigma v_{x_1 x_2} + \sigma^2 v_{x_2 x_2} \right)$$

$$- \left( r - q - \frac{x_2}{2} \right) v_{x_1} - \kappa (\theta - x_2) v_{x_2} + rv, \quad v \in C^\infty(\mathbb{H}).$$

The (constant) coefficients of $A$ obey the ellipticity conditions

$$\sigma \neq 0 \quad \text{and} \quad -1 < \rho < 1,$$

and $\kappa > 0, \theta > 0, q \geq 0$, and $r \geq 0$. 
The variational equations and inequalities defined by the Heston operator and suitable weighted Sobolev spaces, due in part to H. Koch (1999), can be solved numerically using finite-element methods which avoid prescribing a boundary condition along \( \{ x_2 = 0 \} \).

The boundary value and obstacle problems can also be solved numerically using finite-difference methods which avoid prescribing a boundary condition along \( \{ x_2 = 0 \} \).
Figure: Finite-difference solution to the degenerate-elliptic boundary value problem, \( Au = f \) on \( \partial \) and \( u = g \) on \( \partial_1 \partial \).
Figure: Finite-difference solution to the degenerate-elliptic obstacle problem, $\min\{Au - f, u - \psi\} = 0$ a.e. on $\partial\Omega$ and $u = g$ on $\partial_1\Omega$. 
Daskalopoulos-Hamilton Hölder norms
Daskalopoulos-Hamilton Hölder norms I

Before we can state our main results, we shall need to describe the Daskalopoulos-Hamilton family of Hölder norms.

Given \( \alpha \in (0, 1) \) and function \( u \) on an open subset \( U \subset \mathbb{H} \), the Daskalopoulos-Hamilton-Koch Hölder seminorm, \( [u]_{C^\alpha_s(\bar{U})} \), is defined by mimicking the definition of the standard Hölder seminorm, \( [u]_{C^\alpha(\bar{U})} \), except that the usual Euclidean distance between points, \( |x - y| \), is replaced by the distance function, \( s(x, y) \) for \( x, y \in \mathbb{H} \), corresponding to the cycloidal Riemannian metric on \( \mathbb{H} \) defined by

\[
ds^2 = \frac{1}{x_d} \sum_{i=1}^{d} dx_i^2, \tag{4}
\]

where \( x = (x_1, \ldots, x_d) \).
Daskalopoulos-Hamilton Hölder norms II

It is convenient to choose a more explicit *cycloidal distance function* on $\tilde{\mathbb{H}}$ which is equivalent to the distance function defined by the Riemannian metric $(4)$, such as

$$s(x, y) := \frac{|x - y|}{\sqrt{x_d + y_d + |x - y|}}, \quad \forall x, y \in \tilde{\mathbb{H}}. \quad (5)$$

It will be convenient to denote

$$\mathcal{O} := \mathcal{O} \cup \partial_0 \mathcal{O},$$

as distinct from

$$\tilde{\mathcal{O}} = \mathcal{O} \cup \partial \mathcal{O} = \mathcal{O} \cup \partial_0 \mathcal{O} \cup \overline{\partial_1 \mathcal{O}} = \mathcal{O} \cup \overline{\partial_0 \mathcal{O}} \cup \partial_1 \mathcal{O}.$$ 

Daskalopoulos and Hamilton then provide the
Definition ($C^\alpha_s$ norm and Banach space)

Given $\alpha \in (0, 1)$ and an open subset $U \subset \mathbb{H}$, we say that $u \in C^\alpha_s(\bar{U})$ if $u \in C(\bar{U})$ and

$$\|u\|_{C^\alpha_s(\bar{U})} < \infty,$$

where

$$\|u\|_{C^\alpha_s(\bar{U})} := [u]_{C^\alpha_s(\bar{U})} + \|u\|_{C(\bar{U})},$$

(6)

and

$$[u]_{C^\alpha_s(\bar{U})} := \sup_{x,y \in U \atop x \neq y} \frac{|u(x) - u(y)|}{s(x, y)^\alpha}.$$  

(7)

We say that $u \in C^\alpha_s(U)$ if $u \in C^\alpha_s(\bar{V})$ for all precompact open subsets $V \subset U$. 

Daskalopoulos-Hamilton Hölder norms IV

We let $C_{s,loc}^\alpha(\bar{U})$ denote the linear subspace of functions $u \in C_s^\alpha(U)$ such that $u \in C_s^\alpha(\bar{V})$ for every precompact open subset $V \subseteq \bar{U}$.

Definition ($C_s^{1,\alpha}$ norm and Banach space)

Given $\alpha \in (0,1)$ and an open subset $U \subset \mathbb{H}$, we say that $u \in C_s^{1,\alpha}(\bar{U})$ if $u, u_{x_i} \in C_s^\alpha(\bar{U})$ for $1 \leq i \leq d$. We define

$$
\|u\|_{C_s^{1,\alpha}(\bar{U})} := \|u\|_{C_s^\alpha(\bar{U})} + \|Du\|_{C_s^\alpha(\bar{U})}.
$$

We say that $u \in C_s^{1,\alpha}(U)$ if $u \in C_s^{1,\alpha}(\bar{V})$ for all precompact open subsets $V \subseteq U$. 

\[8\]
Daskalopoulos-Hamilton Hölder norms V
Definition ($C^{2+\alpha}_s$ norm and Banach space)

Given $\alpha \in (0, 1)$ and an open subset $U \subset \mathbb{H}$, we say that $u \in C^{2+\alpha}_s(\bar{U})$ if $u \in C^{1,\alpha}_s(\bar{U})$ and the derivatives, $u_{x_ix_j}$, for $1 \leq i, j \leq d$, are continuous on $U$, and the functions, $x_d u_{x_ix_j}$ extend continuously up to the boundary, $\partial U$, and those extensions belong to $C^\alpha_s(\bar{U})$. We define

$$\|u\|_{C^{2+\alpha}_s(\bar{U})} := \|u\|_{C^{1,\alpha}_s(\bar{U})} + \|x_d D^2 u\|_{C^\alpha_s(\bar{U})}. \tag{9}$$

We say that $u \in C^{2+\alpha}_s(U)$ if $u \in C^{2+\alpha}_s(\bar{V})$ for all precompact open subsets $V \Subset U$. 
One can show that if $u \in C^{2+\alpha}_s(U)$, then

$$\chi_d D^2 u = 0 \quad \text{on } \partial_0 U.$$ 

One may similarly define parabolic Daskalopoulos-Hamilton Hölder spaces by modifying the definitions of the standard parabolic Hölder spaces in the analogous way.
Main results
Main results 1

We first consider the question of uniqueness, existence, and regularity of solutions to the elliptic equation with partial Dirichlet boundary condition,

\[ Au = f \quad \text{on } \mathcal{O}, \]  \hspace{1cm} (10)
\[ u = g \quad \text{on } \partial_1 \mathcal{O}. \]  \hspace{1cm} (11)

In the statements of the main theorems, we shall require that the coefficients of \( A \) obey

\[ \langle a \xi, \xi \rangle \geq \lambda_0 |\xi|^2 \quad \text{on } \mathcal{O}, \quad \forall \xi \in \mathbb{R}^d, \]  \hspace{1cm} (12)
\[ b^d > 0 \quad \text{on } \partial_0 \mathcal{O}, \]  \hspace{1cm} (13)
\[ c \geq 0 \quad \text{on } \mathcal{O}, \]  \hspace{1cm} (14)

for some positive constant \( \lambda_0 \).
Main results II

We shall also need to require the coefficients of $A$ should either obey

\[ c \geq c_0 \quad \text{on } \mathcal{O}, \quad (15) \]

or

\[ a^{dd} \leq \Lambda \quad \text{on } \mathcal{O}, \quad (16) \]
\[ b^{d} \geq b_0 \quad \text{on } \mathcal{O}, \quad (17) \]

for some positive constants $b_0$ and $\Lambda$. We then have the
Main results III
Theorem (Existence of a smooth solution to the boundary value problem with partial Dirichlet data)

Let \( \mathcal{O} \subseteq \mathbb{H} \) be a bounded domain such that \( \partial_1 \mathcal{O} \) obeys an exterior sphere condition and \( \alpha \in (0, 1) \). Let \( A \) be as in (1) with coefficients belonging to \( C^\alpha(\mathcal{O}) \), and obeying (12), (14), and (13). Moreover, the coefficients of \( A \) should obey either

1. Condition (15), or
2. Conditions (16) and (17).

If

\[ f \in C^\alpha_s(\mathcal{O}) \cap C_b(\mathcal{O}) \quad \text{and} \quad g \in C_b(\partial_1 \mathcal{O}), \]

then there is a unique solution,

\[ u \in C^{2+\alpha}_s(\mathcal{O}) \cap C_b(\mathcal{O} \cup \partial_1 \mathcal{O}), \]

to the partial Dirichlet boundary value problem (10), (11).
Main results IV

We next consider the question of uniqueness, existence, and regularity of solutions to the obstacle problem,

\[ \min\{Au - f, \ u - \psi\} = 0 \quad \text{a.e. on } \partial, \tag{18} \]

with partial Dirichlet boundary condition (11). We then have the
Main results V

Theorem (Existence of a smooth solution to the obstacle problem with partial Dirichlet data)

Let $\mathcal{O} \subseteq \mathbb{H}$ be a domain such that $\partial_1 \mathcal{O}$ obeys an exterior sphere condition, and $d < p < \infty$, and $\alpha \in (0, 1)$. Assume the hypotheses for $A$, $f$, $g$, and $\mathcal{O}$ in the existence theorem for the boundary value problem. If

$$\psi \in C^2(\mathcal{O}) \cap C(\mathcal{O} \cup \partial_1 \mathcal{O})$$

obeys $\psi \leq g$ on $\partial_1 \mathcal{O}$ and $\sup_{\mathcal{O}} \psi < \infty$, then there is a unique solution,

$$u \in C^{2+\alpha}_{s}(\Omega) \cap W^{2,p}_{\text{loc}}(\mathcal{O}) \cap C^{1,\alpha}_{s}(\mathcal{O}) \cap C_b(\mathcal{O} \cup \partial_1 \mathcal{O}),$$

to the obstacle problem (18), (11), where

$\Omega := \{ x \in \mathcal{O} : (u - \psi)(x) > 0 \}$. 
Main results VI

Remarks (Comments on and extensions to Theorems 4.1 and 4.2)

1. We assumed that $\Omega$ is bounded to simplify the statements here; when $\Omega$ is unbounded, the same conclusions hold provided the coefficients of $A$ obey suitable growth properties.

2. As in the classical case where $A$ is strictly elliptic, the $C^2$ regularity condition on $\psi$ can be relaxed to one where $\psi$ is Lipschitz and obeys a convexity condition.

3. The analogous results hold when the boundary-degenerate elliptic operator, $A$, is replaced by a boundary-degenerate parabolic operator, $L$. 
4. For uniqueness of solutions, the assumption that

$$Au := -x_d \operatorname{tr}(aD^2 u) - \langle b, Du \rangle + cu,$$

where $a \geq \lambda_0 I_d$ on $\partial \Omega$ can be relaxed to

$$Au := -\operatorname{tr}(aD^2 u) - \langle b, Du \rangle + cu,$$

where $a \geq 0$ on $\partial \Omega$ and $a, b, c$ and $u$ obey milder conditions near $\partial_0 \Omega$; stronger regularity conditions on $u$ up to $\partial_0 \Omega$ allowing weaker conditions on $a, b, c$ and vice versa.

5. For existence and regularity of solutions, one needs to make more explicit assumptions on the structure of the coefficients of $D^2 u$ in $A$ than one needs merely to obtain uniqueness.
Maximum principles for boundary-degenerate operators and uniqueness with partial Dirichlet boundary conditions
Maximum principles and uniqueness I

We shall illustrate the essence of weak maximum principle for boundary-degenerate operators and uniqueness with partial Dirichlet boundary conditions with the aid of the homogeneous Kummer ordinary differential equation \cite[§13]{18} on the interval \((0, \ell)\), for \(0 < \ell < \infty\),

\[
Au(x) := -xu''(x) - (b - x)u'(x) + au(x) = 0, \quad x \in (0, \ell),
\]

where \(a \geq 0\) and \(b \geq 0\) are constants.

Equation (19) has two fundamental solutions, \(M(a, b, x)\) and \(U(a, b, x)\), called confluent hypergeometric functions.

It is known that \(M \in C^\infty[0, \infty)\), when \(b > 0\), while \(U \in C[0, \infty)\) and \(U \notin C^1[0, \infty)\) when \(a > 0\) and \(b > 0\).

Therefore, requiring that \(u \in C^2(0, \ell) \cap C^1[0, \ell) \cap C[0, \ell]\) and \(u(\ell) = 0\) ensures that if \(u\) solves (19), then \(u \equiv 0\) on \((0, \ell)\).
Maximum principles and uniqueness II

We recall that

\[ Au = -x_d \text{tr}(aD^2 u) - \langle b, Du \rangle + cu, \]

where the coefficients are merely assumed to be everywhere-defined on \( \mathcal{O} \).

**Theorem (Strong maximum principle)**

Let \( \mathcal{O} \subset \mathbb{R}^d \) be a domain and \( A \) be as in (1). Assume that \( a \) is locally strictly elliptic on \( \mathcal{O} \), \( a \in C(\partial_0 \mathcal{O}; \mathcal{S}^+(d)) \), \( b^d > 0 \) on \( \partial_0 \mathcal{O} \),

\[ b \in C^2(\mathcal{O}; \mathbb{R}^d), \tag{20} \]

\( c^+ \in L_{\text{loc}}^\infty(\mathcal{O}) \), and \( c \geq 0 \) on \( \mathcal{O} \). If \( c = 0 \) (respectively, \( c \geq 0 \)) on \( \mathcal{O} \) and \( u \) attains a global maximum (respectively, non-negative global maximum) in \( \mathcal{O} \), then \( u \) is constant on \( \mathcal{O} \).
Maximum principles and uniqueness III
Theorem (Weak maximum principle)

Let \( \mathcal{O} \subset \mathbb{R}^d \) be a bounded domain and \( A \) be as in (1). Assume that \( a \) is locally strictly elliptic on \( \mathcal{O} \), \( a \in C(\partial_0 \mathcal{O}; \mathcal{L}^+(d)) \), \( b^d > 0 \) on \( \partial_0 \mathcal{O} \),

\[
b \in C(\overline{\mathcal{O}}; \mathbb{R}^d),
\]

(21)

\( c^+ \in L_{\text{loc}}^\infty(\overline{\mathcal{O}}) \), and \( c \geq 0 \) on \( \mathcal{O} \). Suppose \( u \in C^2(\mathcal{O}) \cap C^1(\overline{\mathcal{O}}) \) and \( \sup_{\mathcal{O}} u < \infty \). If \( Au \leq 0 \) on \( \mathcal{O} \) and \( u^* \leq 0 \) on \( \partial_1 \mathcal{O} \), then \( u \leq 0 \) on \( \mathcal{O} \).
Maximum principles and uniqueness IV
Remark (Comments on the maximum principles)

1. Note that points in the degenerate boundary portion, $\partial_0 \mathcal{O}$, play the same role as points in the interior, $\mathcal{O}$, recalling that $\overline{\mathcal{O}} = \mathcal{O} \cup \partial_0 \mathcal{O}$.

2. The maximum principles also hold for $u \in W^{2,d}_{\text{loc}}(\mathcal{O}) \cap C^1(\mathcal{O})$, when $A$ has measurable coefficients and $Au \leq 0$ a.e. on $\mathcal{O}$.

3. The analogous maximum principles also hold for the parabolic operator, $L$.

4. These maximum principles imply uniqueness of solutions to elliptic and parabolic boundary value and obstacle problems with partial Dirichlet boundary data.
We may assume more generally that

\[ Au = - \text{tr}(aD^2 u) - \langle b, Du \rangle + cu, \]

where the coefficients are merely assumed to be everywhere-defined on \( \mathcal{O} \).

**Definition**

We say that \( u \in C^2_s(\mathcal{O}) \) if \( u \in C^2(\mathcal{O}) \cap C^1(\mathcal{O}) \) with \( x_d D^2 u \in C(\mathcal{O}) \) and \( x_d D^2 u = 0 \) on \( \partial_0 \mathcal{O} \).
Maximum principles and uniqueness VI

Theorem (Weak maximum principle)

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded open subset. Require that the coefficients of $A$ be defined everywhere on $\mathcal{O}$ and obey

\begin{align*}
a &\geq 0 \quad \text{on } \mathcal{O}, \quad (22) \\
b^d &\geq 0 \quad \text{on } \partial_0 \mathcal{O}, \quad (23) \\
c &\geq 0 \quad \text{on } \mathcal{O}. \quad (24)
\end{align*}

Assume further that at least one of the following holds:

\[
\begin{cases} 
  c > 0 \quad \text{on } \mathcal{O}, \quad \text{or} \\
  b^d > 0 \quad \text{on } \partial_0 \mathcal{O} \quad \text{and} \quad b^d/a^{dd} \quad \text{locally bounded below on } \mathcal{O}. 
\end{cases} \quad (25)
\]

If $u \in C^2_s(\mathcal{O}) \cap C(\mathcal{O} \cup \partial_1 \mathcal{O})$ obeys $Au \leq 0$ on $\mathcal{O}$ and $u \leq 0$ on $\partial_1 \mathcal{O}$, and $\sup_{\mathcal{O}} u < \infty$, then $u \leq 0$ on $\mathcal{O}$. 
Maximum principles and uniqueness VII

Theorem (Strong maximum principle)

Let $\mathcal{O} \subseteq \mathbb{H}$ be a domain. Assume that the coefficients of $A$ obey

\[
c \geq 0 \quad \text{on } \mathcal{O},
\]

$\begin{align*}
a, b, c & \text{ are locally bounded on } \mathcal{O}, \\
a, b & \text{ are continuous on } \partial_0 \mathcal{O}, \\
a & \text{ is locally strictly elliptic on } \mathcal{O}, \\
b^d & > 0 \quad \text{on } \partial_0 \mathcal{O}.
\end{align*}$

Then one of the following holds.

1. If $c = 0$ on $\mathcal{O}$ and $u$ attains a maximum in $\mathcal{O}$, then $u$ is constant on $\mathcal{O}$.

2. If $c \geq 0$ on $\mathcal{O}$ and $u$ attains a non-negative maximum in $\mathcal{O}$, then $u$ is constant on $\mathcal{O}$. 
Degenerate-elliptic problems with partial Dirichlet boundary conditions and the Perron method
Surprisingly, the Daskalopoulos-Hamilton (or Koch) paradigms do not easily adapt to the case of a boundary value problem, with partial Dirichlet boundary condition,

\[ Au = f \quad \text{on } \partial \Omega, \]
\[ u = g \quad \text{on } \partial_1 \Omega, \]

when \( \partial_1 \subset \neq \partial \Omega \) is non-empty.

In Daskalopoulos and Hamilton (1998) or Koch (1999), who consider parabolic boundary value problems exclusively, one either has \( \Omega = H \) or more generally \( \partial_0 \Omega = \partial \Omega \), but not \( \partial_0 \Omega \subset \neq \partial \Omega \) and so the need to consider ‘corner points’, \( \overline{\partial_0 \Omega \cap \partial_1 \Omega} \), does not arise.

The reasons for difficulties when the set of corner points, \( \overline{\partial_0 \Omega \cap \partial_1 \Omega} \), is non-empty are primarily due to the
Overview of the proof of existence and regularity II

1. Lack (so far) of an a priori global Schauder estimate for a solution $u \in C^{2+\alpha_s}(\bar{\Omega})$ such as,

$$\|u\|_{C^{2+\alpha_s}(\bar{\Omega})} \leq C \left(\|f\|_{C^{\alpha_s}(\bar{\Omega})} + \|g\|_{C^{2+\alpha_s}(\bar{\Omega})}\right).$$

2. Apparent failure of the continuity method due to ‘compatibility conditions’ at the domain corner points, $\partial_0 \Omega \cap \partial_1 \Omega$.

Compatibility issues arising to a Dirichlet boundary condition along $\partial_1 \Omega$ and no boundary condition along $\partial_0 \Omega$ appear to have been first noticed by Kohn and Nirenberg (1967) in the context of boundary-degenerate elliptic problems with partial Dirichlet data.
Overview of the proof of existence and regularity III

We shall prove existence of solutions to both the boundary value and obstacle problems with a partial Dirichlet data using a version of the (classical) Perron method for existence of smooth solutions via the following steps.

1. Develop an a priori ‘interior’ Schauder estimate for a solution $u \in C^{2+\alpha}_s(\Omega)$ (where points in the boundary $\partial_0 \Omega$ are viewed as interior) (F. and Pop, [11]);

2. Solve the Dirichlet problem for a solution $u \in C^{2+\alpha}_s(\bar{S})$ to

   $$Au = f \quad \text{on } S, \quad u = g \quad \text{on } \partial_1 S,$$

   where $S = \mathbb{R}^{d-1} \times (0, \nu)$ is an infinite ‘slab’ and the boundary portions,

   $$\partial_1 S = \mathbb{R}^{d-1} \times \{\nu\} \quad \text{and} \quad \partial_1 S = \mathbb{R}^{d-1} \times \{0\},$$

   do not touch (F. and Pop, [11]).
3. A suitably defined diffeomorphism between the unit half-ball, $B^+ = \{x \in \mathbb{R}^d : |x| < 1, x_d > 0\}$, and the slab, $S = \mathbb{R}^{d-1} \times (0, \pi/2)$, allows us to solve the Dirichlet problem on the unit half-ball, $B^+$, for a solution $u \in C^{2+\alpha}_s(B^+) \cap C_b(B^+ \cup \partial_1 B^+)$ to

$$Au = f \quad \text{on } B^+, \quad u = g \quad \text{on } \partial_1 B^+,$$

given $f \in C^\alpha_s(B^+) \cap C_b(B^+)$ and $g \in C_b(\partial_1 B^+)$. This step provides missing ingredient required for ‘local solvability’ of the Dirichlet problem in the Perron method.

4. Development of the concept of continuous subsolutions and supersolutions, $\mathcal{S}_{f,g}^{\pm}$, to a degenerate-elliptic operator and their properties via the strong maximum principle.
Overview of the proof of existence and regularity V

5. Adaptation of the classical Perron method for solving the Dirichlet problem for a harmonic function or a solution to a boundary value problem for a strictly elliptic operator to show that the Perron function,

\[ u(x) := \sup_{w \in S, x \in \mathcal{O}} w(x), \quad x \in \mathcal{O}, \]

is a solution to the boundary value problem for a degenerate-elliptic operator with a partial Dirichlet boundary condition.
Overview of the proof of existence and regularity VI

The idea is based on a familiar observation from elementary complex analysis that

- A half-disk in the complex plane is conformally equivalent to an infinite strip, and the
- The holomorphic map defining the conformal equivalence identifies harmonic functions on the half-disk with harmonic functions on the strip.
Overview of the proof of existence and regularity VII

Remark (Alternative approaches using viscosity solutions)

One could envisage first proving existence of viscosity solutions to the boundary value and obstacle problems considered here by adapting previous work of Barles [1] and Ishii [16] for existence of viscosity solutions to fully nonlinear boundary value problems with fully nonlinear boundary conditions.

One could then prove the expected Schauder regularity by adapting the methods of [11], provided one can solve the partial Dirichlet problem on a half-ball.

In practice, this approach is less straightforward and less direct than it might appear at first glance. In particular, standard comparison theorems [2] for viscosity solutions do not immediately apply to problems with a partial Dirichlet boundary condition.
Overview of the proof of existence and regularity VIII

Figure: Conformal equivalence between the half-disk, quadrant, and infinite strip in the complex plane.

\[ |z| < 1, \quad \text{Im} \, z > 0 \]

\[ z = -1 \quad \text{to} \quad z = 1 \]

\[ \xi = 0, \quad \text{Re} \, \xi > 0, \quad \text{Im} \, \xi > 0 \]

\[ 0 < \text{Im} \, w < \pi / 2 \]
Overview of the proof of existence and regularity IX

A similar strategy can be also used to prove existence of a solution to the obstacle problem for a degenerate-elliptic operator with a partial Dirichlet boundary condition.

For the obstacle problem, one must also

1. Develop the concept of continuous supersolutions, $\mathcal{S}_{f,g,\psi}^+$, the obstacle problem for a degenerate-elliptic operator and their properties via the strong maximum principle;

2. Prove a comparison principle for a solution and continuous supersolution to the obstacle problem.

3. Adapt the classical Perron method for solving the Dirichlet problem for a harmonic function to show that the Perron function,

$$u(x) := \inf_{w \in \mathcal{S}_{f,g,\psi}^+} w(x), \quad x \in \mathcal{D},$$

is a solution to the obstacle problem for a degenerate-elliptic operator with a partial Dirichlet boundary condition.
Interior Schauder estimates and regularity for solutions to degenerate-elliptic equations
Theorem (A priori Schauder interior estimate and regularity on domains)

For any $\alpha \in (0, 1)$, the following holds. If $u \in C^2(\Omega)$ obeys

$$u \in C^1(\Omega), \quad x_d D^2 u \in C(\Omega), \quad Au \in C_\ast^\alpha(\Omega),$$

and

$$x_d D^2 u = 0 \text{ on } \partial_0 \Omega,$$

then $u \in C_{\ast}^{2+\alpha}(\Omega)$. Moreover, for each subdomain $\Omega' \subset \Omega$ with $\tilde{\Omega}' \subset \tilde{\Omega}$, we have $u \in C_{\ast}^{2+\alpha}(\tilde{\Omega}')$ and

$$\|u\|_{C_{\ast}^{2+\alpha}(\tilde{\Omega}')} \leq C \left(\|u\|_{C(\tilde{\Omega})} + \|Au\|_{C_\ast^\alpha(\tilde{\Omega})}\right).$$
Global Schauder a priori estimates and existence of solutions on the infinite slab
An a priori estimate

It appears considerably more difficult to prove a global a priori estimate for a solution, \( u \in C^{k, 2+\alpha}_{s}(\bar{\Omega}) \), when the intersection \( \partial_0 \Omega \cap \partial_1 \Omega \) is non-empty.

The reason, in part, is that when a domain has corners, solutions to (homogeneous) Dirichlet boundary value problems may have singularities at the corner points.

However, a global estimate on an infinite horizontal slab has useful applications when \( \partial_1 \Omega \) does not meet \( \partial_0 \Omega \).
A priori global Schauder estimate on a slab

Theorem (A priori global Schauder estimate on a slab)

For any $\alpha \in (0, 1)$, and positive constants $\lambda_0$, $b_0$, $\Lambda$, $\nu$, and integer $k \geq 0$, there is a positive constant, $C = C(k, \alpha, \nu, d, \lambda_0, b_0, \Lambda)$, such that the following holds. Suppose the coefficients of $A$ in (1) belong to $C_s^{k, \alpha}(\bar{S})$, where $S = \mathbb{R}^{d-1} \times (0, \nu)$, and obey

$$\|a\|_{C_s^{k, \alpha}(\bar{S})} + \|b\|_{C_s^{k, \alpha}(\bar{S})} + \|c\|_{C_s^{\alpha}(\bar{S})} \leq \Lambda,$$  \hspace{1cm} (29)

$$\langle a\xi, \xi \rangle \geq \lambda_0|\xi|^2 \quad \text{on} \ \bar{S}, \quad \forall \xi \in \mathbb{R}^d,$$  \hspace{1cm} (30)

$$b^d \geq b_0 \quad \text{on} \ \partial_0 S.$$  \hspace{1cm} (31)

If $u \in C_s^{k, 2+\alpha}(\bar{S})$ and $u = 0$ on $\partial_1 S$, then

$$\|u\|_{C_s^{k, 2+\alpha}(\bar{S})} \leq C \left(\|Au\|_{C_s^{k, \alpha}(\bar{S})} + \|u\|_{C(\bar{S})}\right).$$  \hspace{1cm} (32)
Existence and uniqueness of smooth solutions on a slab

Theorem (Existence and uniqueness of a $C_s^{k,2+\alpha}(\bar{S})$ solution on a slab $S$)

Let $\alpha \in (0,1)$, let $\nu > 0$ and $S = \mathbb{R}^{d-1} \times (0,\nu)$, and let $k \geq 0$ be an integer. Let $A$ be an operator as in (1). If $f$ and the coefficients of $A$ in (1) belong to $C_s^{k,\alpha}(\bar{S})$ and obey (30) and (31) for some positive constants $b_0, \lambda_0$, then there is a unique solution, $u \in C_s^{k,2+\alpha}(\bar{S})$, to the boundary value problem,

$$Au = f \quad \text{on } S,$$

$$u = 0 \quad \text{on } \partial_1 S. \quad (33)$$

If $g \in C_s^{k,2+\alpha}(\bar{S})$, then the homogeneous Dirichlet condition, $u = 0$ on $\partial_1 S$, may be replaced by $u = g$ on $\partial_1 S$. 

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Corollary (Existence and uniqueness of a $C_s^{k,2+\alpha}(S) \cap C(\bar{S})$ solution to the inhomogeneous Dirichlet problem on a slab $S$)  

Let $\alpha \in (0,1)$, let $\nu > 0$ and $S = \mathbb{R}^{d-1} \times (0,\nu)$, and let $k \geq 0$ be an integer. Let $A$ be an operator as in (1). If $f$ and the coefficients of $A$ in (1) belong to $C_s^{k,\alpha}(S)$ and obey (30) and (31) for some positive constants $b_0, \lambda_0$ and $g \in C(\partial_1 S)$, then there is a unique solution, $u \in C_s^{k,2+\alpha}(S) \cap C(\bar{S})$, to the boundary value problem,

$$Au = f \quad \text{on } S, \quad (35)$$
$$u = g \quad \text{on } \partial_1 S. \quad (36)$$
Compatibility at the corner points
Compatibility at the corner points I

When \( \partial_0 \Omega \cap \partial_1 \Omega \) is non-empty, it appears to be a challenging problem to prove existence of solutions, \( u \), in \( C^{k,2+\alpha}_s(\Omega) \cap C(\bar{\Omega}) \) or \( C^{k,2+\alpha}_s(\bar{\Omega}) \) to the elliptic boundary value problem,

\[
Au = f \quad \text{on} \quad \partial \Omega, \tag{37}
\]
\[
u = g \quad \text{on} \quad \partial_1 \Omega. \tag{38}
\]

For example, complications emerge when one attempts to apply the continuity method to prove existence of solutions \( u \in C^{k,2+\alpha}_s(\bar{\Omega}) \) to (37), (38), even in the simplest case, \( g = 0 \) on \( \partial_1 \Omega \).

While the reflection principle (across the axis \( x = 0 \)) does not hold for the Heston operator,

\[
Av := -\frac{y}{2} \left( v_{xx} + 2\varrho \sigma v_{xy} + \sigma^2 v_{yy} \right) - \left( c_0 - q - \frac{y}{2} \right) v_x - \kappa (\theta - y) v_y + c_0 v,
\]
it does hold for the simpler model operator,

\[ A_0 v := -\frac{y}{2} (v_{xx} + \sigma^2 v_{yy}) - \kappa (\theta - y) v_y + c_0 v, \quad v \in C^\infty(\mathbb{H}), \]

since the \( v_{xy} \) and \( v_x \) terms are absent. If \( f(-x, y) = -f(x, y) \) for \( (x, y) \in \mathbb{H} \) (and thus \( f(0, \cdot) = 0 \)), one can solve

\[ A_0 u_0 = f \quad \text{on} \quad \partial, \quad u = 0 \quad \text{on} \quad \partial_1 \partial, \]

for a solution, \( u_0 \), when the domain, \( \partial \), is the quadrant \( \mathbb{R}_+ \times \mathbb{R}_+ \).

The continuity method would proceed by showing that, given \( f \in C^\alpha_s(\partial) \), the set of \( t \in [0, 1] \) such that

\[ A_t u = f \quad \text{on} \quad \partial, \quad u = g \quad \text{on} \quad \partial_1 \partial, \]
Compatibility at the corner points III

has a solution $u \in C^{2+\alpha}_s(\bar{\Omega})$ is non-empty, open, and closed, where

$$A_t v := \frac{-y}{2} (v_{xx} + 2t\sigma v_{xy} + \sigma^2 v_{yy}) - t\left(c_0 - q - \frac{y}{2}\right)v_x$$

$$- \kappa(\theta - y)v_y + c_0 v, \quad v \in C^\infty(\mathbb{H}),$$

is a family of operators, $C^{2+\alpha}_s(\bar{\Omega}) \to C^\alpha_s(\bar{\Omega})$, connecting $A_0$ to $A$.

However, if $u \in C^{2+\alpha}_s(\bar{\Omega})$ solves $A_t u = f$ on $\partial\Omega$, $u = 0$ on $\partial_1\partial\Omega$, then, letting $y \to 0$, we find

$$-t(c_0 - q)u_x(0, 0) = f(0, 0),$$

since $u_y(0, 0) = 0$ (because $u(0, \cdot) = 0$) and as $u \in C^{2+\alpha}_s(\bar{\Omega})$ implies $\lim_{(x, y) \to (0, 0)} yD^2 u = 0$.

When $t(c_0 - q) = 0$, we see that we can only solve $A_0 u = f$ on $\partial\Omega$, $u = 0$ on $\partial_1\partial\Omega$ when $f$ obeys the compatibility condition $f(0, 0) = 0$, but this is not present when $t(c_0 - q) \neq 0$. 
Existence and uniqueness of smooth solutions to the inhomogeneous partial Dirichlet problem on a half-ball
Existence of solutions to the inhomogeneous partial Dirichlet problem on a half-ball

Suppose $\mathcal{O} = B^+ = \{x \in \mathbb{R}^d : |x| < 1, \; x_d > 0\}$, so that $\partial_0 B^+ \cap \partial_0 B^+ = \{x \in \mathbb{R}^d : |x| = 1, \; x_d = 0\}$, where $\partial_0 B^+$ is the flat boundary (in $\{x_d = 0\}$) and $\partial_1 B^+$ is the curved boundary (in $\{x_d > 0\}$). Despite the preceding difficulties, one can prove

Theorem (Existence of solutions to the partial Dirichlet problem on a half-ball)

Let $\alpha \in (0,1)$ and let $k \geq 0$ be an integer. Let $A$ be an operator as in (1). If $f$ and the coefficients of $A$ in (1) belong to $C^{k,\alpha}_s(B^+)$ and obey (30) and (31) for some positive constants $b_0, \lambda_0$ and $g \in C_b(\partial_1 B^+)$, then there is a unique solution, $u \in C^{k,2+\alpha}_s(B^+) \cap C_b(B^+ \cup \partial_1 B^+)$, to the boundary value problem,

$$Au = f \quad \text{on} \; B^+, \quad (39)$$
$$u = g \quad \text{on} \; \partial_1 B^+. \quad (40)$$
Outline of the proof of the theorem in dimension two

When \( d = 2 \), the proof is simpler as we may use methods of complex analysis and prove the theorem via the following strategy:

1. Apply a conformal map to transform the half-ball \( B^+ \) to the infinite strip \( S = \mathbb{R} \times (0, \pi/2) \).
2. Transform the boundary value problem on the half-ball,
   \[
   Au = f \quad \text{on } B^+, \quad u = g \quad \text{on } \partial_1 B^+,
   \]
to an equivalent Dirichlet boundary value problem on the strip,
   \[
   \tilde{A}\tilde{u} = \tilde{f} \quad \text{on } S, \quad \tilde{u} = g \quad \text{on } \partial_1 S.
   \]
3. Apply a limit argument to address the fact that \( \tilde{f} \) and the coefficients \( \tilde{b} \) and \( \tilde{c} \) of \( \tilde{A} \) become unbounded as \( |s| \to \infty \), where \( (s, \theta) \in S \).
A half-ball in the complex plane

Using $z = x + iy \in \mathbb{C}$, the conformal transformation,

$$\mathbb{C} \ni z \mapsto \xi = \frac{1 + z}{1 - z} \in \mathbb{C}$$

maps the half-ball ...

Figure: Half-ball, \( \{z \in \mathbb{C} : |z| \leq 1 \text{ and } \Re(z) \geq 0 \} \), in the complex plane.
A quadrant in the complex plane

... onto the quadrant, and the conformal transformation,

\[ C \ni \xi \mapsto \text{Log} \xi = \ln |\xi| + i \text{Arg} \xi \in C, \]

maps the quadrant ...

Figure: Quadrant, \( \{z \in \mathbb{C} : \Re z \geq 0 \text{ and } \Im z \geq 0 \} \), in the complex plane.
A horizontal strip in the complex plane

... onto the infinite horizontal strip,

Figure: Strip, \( \{ z \in \mathbb{C} : 0 \leq \Im z \leq \pi/2 \} \), in the complex plane.
Conformal map from the half-ball to the strip

The composite transformation,

\[ \mathbb{C} \ni z \mapsto w = \log \left( \frac{1 + z}{1 - z} \right) = s + i\theta \in \mathbb{C}, \]

maps the

- **Semicircle**, \( \{ z \in \mathbb{C} : |z| = 1 \text{ and } \Re z > 0 \} \), onto the line, \( \{ z \in \mathbb{C} : \Im z = \pi/2 \} \),
- **Interval**, \( \{ z \in \mathbb{C} : |\Re z| < 1 \text{ and } \Im z = 0 \} \), onto the line, \( \{ z \in \mathbb{C} : \Im z = 0 \} \),
- **Points** \( \{ \pm 1 \} \) to \( \infty \in \mathbb{C} \cup \{ \infty \} \).
Figure: Image of a finite rectangular strip, \( \{w \in \mathbb{C} : -a \leq \text{Re } w \leq a, 0 \leq \text{Im } w \leq \pi/2\} \), of width \( 2a > 0 \) in the complex \( z \)-plane under the conformal map \( w \mapsto z = (e^w - 1)/(e^w + 1) \) from the infinite strip, \( \{w \in \mathbb{C} : 0 \leq \text{Im } w \leq \pi/2\} \), in the complex \( w \)-plane to the unit half-disk in the complex \( z \)-plane without the corner points, \( \{z \in \mathbb{C} : |z| \leq 1, \text{Im } z \geq 0, z \neq \pm 1\} \).
Transformation of the operator \( I \)

We shall illustrate the transformation when \( A \) has constant coefficients and has the form

\[
Au := -y (u_{xx} + u_{yy}) - b_1 u_x - b_2 u_y + cu = f.
\]

We recall the effect on Laplace operator, \( \Delta_z u(x, y) = u_{xx} + u_{yy} \).

Writing \( u(x, y) = u(z) = v(w) = v(s, \theta) \) and \( \Delta_w v(s, \theta) = v_{ss} + v_{\theta\theta} \), we have

\[
\Delta_z u = \frac{\partial^2 u}{\partial z \partial \bar{z}} = \left| \frac{\partial w}{\partial z} \right|^2 \Delta_w v = 2 \frac{|1 - z|^2}{|1 + z|^6} \Delta_w v,
\]

where

\[
\Delta_w v = \frac{\partial^2 v}{\partial w \partial \bar{w}}.
\]

A calculation shows that \( Au = f \) on \( B^+ \) if and only if

\[
\tilde{A}v := -\theta (v_{ss} + v_{\theta\theta}) - \tilde{b}_1 v_s - \tilde{b}_2 v_\theta + \tilde{c}v = \tilde{f} \quad \text{on} \ S,
\]
Transformation of the operator II

where

\[\tilde{b}_1 := \frac{4\theta e^{4s}}{\sin \theta (1 + 2e^s \cos \theta + e^{2s})} \left[ b_1 \left( \frac{\cos \theta}{2} (1 + 2e^s \cos \theta + e^{2s}) + \sin^2 \theta \right) \right.\]
\[- b_2 \left. \sin \theta \left( e^{2s} - 1 \right) \right] , \]

and

\[\tilde{b}_2 := \frac{4\theta e^{4s}}{\sin \theta (1 + 2e^s \cos \theta + e^{2s})} \left[ b_1 \frac{\sin \theta}{2} (e^{2s} - 1) \right.\]
\[+ b_2 \left( \frac{\cos \theta}{2} (1 + 2e^s \cos \theta + e^{2s}) + \sin^2 \theta \right) \right] , \]
Transformation of the operator III

and

\[ \tilde{c} := \frac{4c\theta e^{4s}}{\sin \theta (1 + 2e^s \cos \theta + e^{2s})} , \]

\[ \tilde{f} := \frac{4f\theta e^{4s}}{\sin \theta (1 + 2e^s \cos \theta + e^{2s})} . \]

The coefficients extend continuously from \((s, \theta) \in \mathbb{R} \times (0, \pi/2]\) to \((s, \theta) \in \mathbb{R} \times [0, \pi/2]\). Indeed, for any fixed \(s \in \mathbb{R}\), we obtain

\[ \lim_{\theta \downarrow 0} \tilde{b}_1(s, \theta) = 2e^{4s} b_1 \quad \text{and} \quad \lim_{\theta \downarrow 0} \tilde{b}_2(s, \theta) = 2e^{4s} b_2, \]

\[ \lim_{\theta \downarrow 0} \tilde{c}(s, \theta) = \frac{4ce^{4s}}{(1 + e^s)^2} \quad \text{and} \quad \lim_{\theta \downarrow 0} \tilde{f}(s, \theta) = \frac{4fe^{4s}}{(1 + e^s)^2}, \quad s \in \mathbb{R}. \]

Note that \(b_2 \geq 0\) if and only if \(\tilde{b}_2(s, 0) \geq 0\) and \(b_2 > 0\) if and only if \(\tilde{b}_2(s, 0) > 0\).
Continuous subsolutions and supersolutions and the Perron method
Continuous subsolutions and supersolutions I

By analogy with the definitions of continuous subharmonic and superharmonic functions or continuous subsolutions and supersolutions to linear, second-order elliptic partial differential equations, we make the

Definition. (Continuous subsolution and supersolution to an elliptic equation and boundary problem) Let \( \mathcal{O} \subseteq \mathbb{H} \) be a domain and \( A \) be a degenerate-elliptic operator. Given \( f \in C(\mathcal{O}) \), we call \( u : \mathcal{O} \to \mathbb{R} \) a continuous subsolution to the equation,

\[
Aw = f \quad \text{on} \quad \mathcal{O},
\]

if \( u \) is continuous on \( \mathcal{O} \), locally bounded on \( \mathcal{O} \), and for every open ball \( B \subseteq \mathcal{O} \) or half-ball \( B^+ \subseteq \mathcal{O} \) with center in \( \partial_0 \mathcal{O} \) and for every \( \bar{u} \in C^2(U) \cap C^1(U) \), with \( U = B \) or \( B^+ \), obeying

\[
xdD^2\bar{u} \in C(U) \quad \text{and} \quad xdD^2\bar{u} = 0 \quad \text{on} \ \partial_0 U,
\]
Continuous subsolutions and supersolutions II

and \( \inf_U \bar{u} > -\infty \), and

\[
\begin{aligned}
A\bar{u} &\geq f \quad \text{on } U, \\
\bar{u} &\geq u \quad \text{on } \partial_1 U,
\end{aligned}
\]

we then have

\[ u \leq \bar{u} \quad \text{on } U. \]

Given \( g \in C(\partial_1 \mathcal{O}) \), we call \( u \in C(\mathcal{O} \cup \partial_1 \mathcal{O}) \) a continuous subsolution to the boundary value problem,

\[
A w = f \quad \text{on } \mathcal{O}, \quad w = g \quad \text{on } \partial_1 \mathcal{O},
\]

if \( u \) is a continuous subsolution to the equation and \( u \leq g \) on \( \partial_1 \mathcal{O} \).

We call \( v \in C(\mathcal{O}) \) a continuous supersolution to the equation,

\[
A w = f \quad \text{on } \mathcal{O},
\]
Continuous subsolutions and supersolutions III

if $-v$ is a subsolution to this equation; we call $v \in C(\Omega \cup \partial_1 \Omega)$ a continuous supersolution to the boundary value problem,

$$Aw = f \quad \text{on } \Omega, \quad w = g \quad \text{on } \partial_1 \Omega,$$

if $-v$ is a continuous subsolution to this boundary value problem.

Remark (Continuity of subsolutions and supersolutions)

It is important to note that in our definition, subsolutions and supersolutions are defined to only be continuous on $\Omega$ and not $\overline{\Omega}$, since continuity is not necessarily preserved along $\partial_0 \Omega$ by the ‘harmonic lifting’ process. Fortunately, this has no impact on the application of Perron’s method.
Continuous subsolutions and supersolutions obey a comparison principle,

\[ u \leq v \quad \text{on } \partial_1 \Omega \implies u \leq v \quad \text{on } \Omega, \]

provided \( A \) obeys some technical conditions.

The technical conditions on \( A \) (and the comparison principle) will again ensure that

- The set of continuous supersolutions to the obstacle problem, \( \mathcal{S}_{f,g}^+ \), is non-empty;
- Every continuous supersolution, \( v \in \mathcal{S}_{f,g}^+ \), is bounded below by a constant \( M \), depending at most on \( f, g \), and one or more of the coefficients of \( A \).
Continuous subsolutions and supersolutions V

Definition. (Solution to an obstacle problem) Let \( \mathcal{O} \subseteq \mathbb{H} \) be a domain, \( 2 < p < \infty \), and \( A \) be a degenerate-elliptic operator. Given \( f \in C(\mathcal{O}) \) and \( \psi \in C(\mathcal{O}) \), we call \( u \in W^{2,p}_{\text{loc}}(\mathcal{O}) \cap C(\mathcal{O}) \) a solution to the elliptic obstacle problem if \( u \) obeys

\[
\min\{Au - f, u - \psi\} = 0 \quad \text{a.e. on } \mathcal{O},
\]

and if \( \Omega = \{x \in \mathcal{O} : (u - \psi)(x) > 0\} \), then \( u \in C^2(\Omega) \cap C^1(\Omega) \) and obeys

\[
x_d D^2 u \in C(\Omega) \quad \text{and} \quad x_d D^2 u = 0 \quad \text{on } \partial_0 \Omega.
\]

Given \( g \in C(\partial_1 \mathcal{O}) \) and \( \psi \) also belonging to \( C(\mathcal{O} \cup \partial_1 \mathcal{O}) \) and obeying the compatibility condition \( \psi \leq g \) on \( \partial_1 \mathcal{O} \), we call \( u \) a solution to the elliptic obstacle problem with Dirichlet boundary condition if \( u \) also belongs to \( C(\mathcal{O} \cup \partial_1 \mathcal{O}) \) and obeys

\[
u = g \quad \text{on } \partial_1 \mathcal{O}.
\]
Definition. (Continuous supersolution to an elliptic obstacle problem) Let \( \mathcal{O} \subseteq \mathbb{R} \) be a domain and \( A \) be a degenerate-elliptic operator. Given \( f \in C(\mathcal{O}) \) and \( \psi \in C(\mathcal{O}) \), we call \( v : \mathcal{O} \rightarrow \mathbb{R} \) a continuous supersolution to the elliptic obstacle problem, if \( v \) is continuous on \( \mathcal{O} \), locally bounded on \( \mathcal{O} \), satisfies
\[
\min\{Aw - f, w - \psi\} = 0 \quad \text{a.e. on} \ \mathcal{O},
\]
and is a continuous supersolution to the elliptic equation
\[
Aw = f \quad \text{on} \ \mathcal{O},
\]
Continuous subsolutions and supersolutions VII
in the sense we defined before: for every open ball $B \in \mathcal{O}$ or half-ball $B^+ \in \mathcal{O}$ and for every $\nu \in C^2(U) \cap C^1(U)$, with $U = B$ or $B^+$, obeying

$$x_d D^2 \nu \in C(U) \quad \text{and} \quad x_d D^2 \nu = 0 \quad \text{on } \partial_0 U,$$

and $\sup U \nu < \infty$, and

$$\left\{
\begin{align*}
Av & \leq f \quad \text{on } U, \\
\nu & \leq v \quad \text{on } \partial_1 U,
\end{align*}\right.$$

we then have

$$\nu \geq v \quad \text{on } U.$$

Given $g \in C(\partial_1 \mathcal{O})$ and $\psi$ also belonging to $C(\mathcal{O} \cup \partial_1 \mathcal{O})$ and obeying $\psi \leq g$ on $\partial_1 \mathcal{O}$, we call $v \in C(\mathcal{O} \cup \partial_1 \mathcal{O})$ a continuous supersolution to the elliptic obstacle problem

$$\min\{A w - f, w - \psi\} = 0 \quad \text{a.e. on } \mathcal{O}, \quad w = g \quad \text{on } \partial_1 \mathcal{O},$$
Continuous subsolutions and supersolutions VIII

if \( v \) is a continuous supersolution to \( Aw = f \) on \( \partial \) and

\[
v \geq g \quad \text{on } \partial_1 \partial.
\]

Solutions and continuous supersolutions obey a comparison principle,

\[
u \leq v \quad \text{on } \partial_1 \partial \implies u \leq v \quad \text{on } \partial,
\]

again provided \( A \) obeys a few additional technical conditions.

The technical conditions on \( A \) (and the comparison principle) will again ensure that

- The set of continuous supersolutions to the obstacle problem, \( \mathcal{S}_{f,g,\psi}^+ \), is non-empty;
- Every continuous supersolution, \( v \in \mathcal{S}_{f,g,\psi}^+ \), is bounded below by a constant \( M \), depending at most on \( f, g, \psi \), and one or more of the coefficients of \( A \).
Modifications required for solution to the degenerate-parabolic terminal-boundary value and obstacle problems with partial Dirichlet boundary conditions
It is possible to extend the strategy previously outlined in the elliptic case to cover terminal-boundary value and obstacle problems with partial Dirichlet boundary conditions for parabolic operators with the same kind of degeneracy.

As in the elliptic case, the following ingredients play a key role:

1. Weak maximum principle for degenerate-parabolic operators and partial boundary comparison (Daskalopoulos and Hamilton [5]; F. and Pop [10]);

2. Strong maximum principle for degenerate-parabolic operators;

3. Schauder a priori interior estimates and regularity for degenerate-parabolic operators (Daskalopoulos and Hamilton [5]; F. and Pop [10]);
Solution to degenerate-parabolic problems II

4. Solution to the Dirichlet problem on an infinite cylindrical slab with global Schauder a priori interior estimate and regularity (Daskalopoulos and Hamilton [5]; F. and Pop [10]);

5. Solution to the Dirichlet problem on a cylindrical half-ball with partial Dirichlet boundary condition;

6. Definition and properties of continuous subsolutions and supersolutions to degenerate-parabolic terminal-boundary value problems and continuous supersolutions to obstacle problems, both with partial Dirichlet boundary conditions.

Solution to degenerate-parabolic problems III

As in the elliptic case, the strong maximum principle plays an essential role in developing the maximum and comparison principles for continuous subsolutions and supersolutions and their application in the Perron method.

We shall consider an operator of the form

$$Lv := -v_t + Av \quad \text{on} \ Q, \quad v \in C^\infty(Q),$$

(41)

where $Q \subset \mathbb{R}^{d+1}$ is an open subset and now

$$Av := -x_d \operatorname{tr}(aD^2v) - \langle b, Dv \rangle + cv \quad \text{on} \ Q, \quad v \in C^\infty(Q).$$

(42)

We will assume a cylindrical domain, $Q = (0, T) \times \mathcal{O}$ for $T > 0$, to simplify further statements, though this is not required.

One needs an analogue of the classical strong maximum principle for linear, strictly parabolic second-order operators due to Nirenberg [17] and refinements due to Friedman [12, 13]
Solution to degenerate-parabolic problems IV

Definition

We say that $u \in C^2_s(Q)$ if $u \in C^2(Q) \cap C^1(Q)$ with $x_d D^2 u \in C(Q)$ and $x_d D^2 u = 0$ on

$$\phi_0 Q := (0, T) \times \partial_0 \mathcal{O} \subset \partial Q.$$ 

Here, $u \in C^2(Q)$ means $D^2 u \in C(Q)$ and $u_t \in C(Q)$. 
Solution to degenerate-parabolic problems V

Theorem (Strong maximum principle when $c \geq 0$)

Let $\Omega \subset \mathbb{R}^d$ be a domain and $Q = (0, T) \times \Omega$ for $T > 0$, and assume that the coefficients of $L$ obey

$$c \geq 0 \quad \text{on} \quad Q,$$
$$a, \ b, \ c \quad \text{are locally bounded on} \quad Q,$$
$$a, b \quad \text{are continuous on} \quad \partial \Omega Q,$$
$$a \quad \text{is locally strictly elliptic on} \quad Q,$$
$$\langle b, \tilde{n} \rangle > 0 \quad \text{on} \quad \partial \Omega Q,$$

where $n_0 e_0 + \tilde{n} = \tilde{n}$ denotes the inward-pointing normal vector field along $\partial \Omega Q$.

If $u \in C^2_s(Q)$ obeys $Lu \leq 0$ on $Q$ and $u$ has a positive maximum at a point $P^0 = (t^0, x^0) \in Q$, then $u = u(P^0)$ on $(0, t^0] \times \Omega$. 


As in the elliptic case, a key point is to prove a type of Hopf boundary-point lemma.

**Proposition (Hopf-type lemma)**

Let $Q \subset \mathbb{R}^{d+1}$ be an open subset and assume that the coefficients of $L$ obey the hypotheses of Theorem 12.2. Assume $u \in C^2_s(Q)$ obeys $Lu \leq 0$ on $Q$ and that $u$ achieves a positive maximum $M$ in $Q$. Suppose that $Q$ contains the closure $\overline{E}$ of an open solid ellipsoid,

$$E := \left\{(t, x) \in \mathbb{R}^{d+1} : \gamma_0 (t - t^*)^2 + \sum_{i=1}^{d} \gamma_i (x_i - x_i^*)^2 < R^2 \right\},$$

where $\gamma_i > 0$ for $i = 0, 1, \ldots, d$, and $R > 0$, and that $u < M$ on $E$ and $u(\bar{t}, \bar{x}) = M$ at some point $(\bar{t}, \bar{x}) \in \partial E$. Then $\bar{x} = x^*$. 
In the classical strictly parabolic (and elliptic) case, one proves the result with the aid of a suitable barrier function, \( h : Q \to \mathbb{R} \), with the properties that

\[
\begin{align*}
    h > 0 & \quad \text{on } E, \\
    h = 0 & \quad \text{on } \partial E, \\
    h < 0 & \quad \text{on } Q \setminus \bar{E}, \\
    Lh < 0 & \quad \text{on } B_{\rho}(\bar{P}),
\end{align*}
\]

for some \( 0 < \rho < |x^* - \bar{x}| \), if \( x^* \neq \bar{x} \). Friedman’s choice is

\[
h(t, x) := \exp \left\{ -\alpha \left[ \gamma_0(t - t^*)^2 + \sum_{i=1}^{d} \gamma_i(x_i - x_i^*)^2 \right] \right\} - \exp(\alpha R^2),
\]

for a large-enough constant \( \alpha > 0 \). (This is a generalization of a similar choice in the proof of the elliptic Hopf boundary point lemma found in Gilbarg and Trudinger [14].)
The standard proofs make essential use of the fact that $A$ and $L$ are strictly elliptic and parabolic, respectively and this is fine when $\bar{P} \in Q$. However, it’s unclear how to choose a barrier function when those conditions fail, as they do for points in $\bar{P} \in \partial_0 Q$.

Nevertheless, an exceedingly simple choice works (also in the classical case) if we first modify the geometry near $\bar{P}$. 
Boundary ball in $\mathbb{R}^{d+1}$

Figure: Boundary ball in $(t, x)$-space, $\mathbb{R}^{d+1}$. 
Boundary ball in $\mathbb{R}^{d+1}$ after ‘downward push’

Figure: Boundary ball in $(t, x)$-space, $\mathbb{R}^{d+1}$, after ‘downward push’.
One now finds that
\[ h(t, x) = x_d, \]
has the required properties,

\[ h > 0 \quad \text{when} \quad x_d > 0, \]
\[ h = 0 \quad \text{when} \quad x_d = 0, \]
\[ h < 0 \quad \text{when} \quad x_d < 0, \]
\[ Lh < 0 \quad \text{on} \quad B_\rho(O) \cap Q, \]

after checking the effect of the diffeomorphism on the coefficients of \( L \).
Beyond the porous medium and Heston equations: other degenerate-elliptic and degenerate-parabolic operators in mathematical finance
Other degenerate operators in mathematical finance I

The following stochastic process model — proposed in 2002 by Hagan, Kumar, Lesniewski, and Woodward [15] — has become very widely used for interest rate derivative modeling:

\[
d \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} Y_1^\beta(s) Y_2(s) & 0 \\ \rho \alpha Y_2(s) & \sqrt{1 - \rho^2} \alpha Y_2(s) \end{pmatrix} d \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix},
\]

where \( W(s) = (W_1(s), W_2(s)) \) is two-dimensional Brownian motion, \( \alpha > 0 \) and \( 0 < \beta < 1 \) are constants, and \( \rho \in (-1, 1) \).

This process has Markov generator, \(-A\), on the quadrant, \( \mathbb{Q} = \mathbb{R}_+ \times \mathbb{R}_+ \),

\[
A \nu = -\frac{y_2^2}{2} \left( y_1^{2\beta} \nu_{y_1y_1} + 2\rho \alpha y_1^\beta \nu_{y_1y_2} + \alpha^2 \nu_{y_2y_2} \right),
\]

where \( \nu \in C^\infty(\mathbb{Q}) \).
Other degenerate operators in mathematical finance II

In the coordinates $X_1(s) := Y_1(s)$ and $X_2(s) := \log Y_2(s)$, the process becomes

$$
d \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \, ds - \begin{pmatrix} X_1^\beta(s) e^{X_2(s)} & 0 \\ \rho \alpha & \sqrt{1 - \rho^2 \alpha} \end{pmatrix},
$$

with Markov generator, $-A$, on the half-plane, $\mathbb{H} = \mathbb{R}_+ \times \mathbb{R}$,

$$
Au = -\frac{1}{2} \left( x_1^{2\beta} e^{2x_2} u_{x_1x_1} + 2\rho \alpha x_1^\beta e^{x_2} u_{x_1x_2} + \alpha^2 u_{x_2x_2} \right) + \frac{\alpha^2}{2} u_{x_2},
$$

where $u \in C^\infty(\mathbb{H})$ and $u(x_1, x_2) = \nu(y_1, \log y_2)$.

This is degenerate in a more complicated way than the Heston operator (both as $x_1 \downarrow 0$ and $x_2 \downarrow -\infty$) and solution of the four problems discussed so far in this presentation appears to require different methods.
Thank you for your attention!


References II


References III


A. Friedman, Remarks on the maximum principle for parabolic equations and its applications, Pacific J. Math. 8 (1958), 201–211. MR 0102655 (21 #1444)
References IV


