Note on
Legendre transform and line-irreducible graphs

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1 Line-irreducible graphs

Definitions. A bridge in a connected graph is an edge whose removal produces a disconnected graph. A line-irreducible graph is a bridgeless connected graph. The term 2-edge-connected graph is also used in this sense. Following [2], a maximal line irreducible subgraph of a graph is called a lump.

Let \( M = M(X) \) be a given class of line-irreducible graphs (called \( M \)-graphs for short) and let \( C_M(X) \) denote the corresponding species of connected graphs all of whose lumps are in \( M \) (called \( C_M \)-graphs). For example, if \( M = X \), the single vertex graph, then \( C_M = a \), the species of trees (\( a \) for French "arbres"). On the other hand, if \( M = M_{\text{all}} \), the class of all line-irreducible graphs, then \( C_M = C \), the species of all connected graphs. Other interesting examples arise by taking \( M \) to be the class of polygons (unoriented cycles), of complete graphs (except \( K_2 \)), or of oriented cycles.

Let \( M^\bullet \) and \( C_M^\bullet \) denote the species of vertex-rooted \( M \)-graphs and \( C_M \)-graphs, respectively. The following result is well known. See for example [1], [2], and [3].

Proposition 1 We have the following isomorphism of species:

\[
C_M^\bullet = M^\bullet(X E(C_M^\bullet)),
\]

where \( E \) denotes the species of sets.

Proof. See Figure 1. Here we have taken \( M = X + K_3 \).
Our goal is to express this relationship in terms of Legendre Transform. Before doing this, it is worthwhile to state the following Dissymmetry Theorem which is less well known, although given in [2] in its cycle index series form for the purpose of unlabelled enumeration. It gives an expression of the species of unrooted $C_M$-graphs in terms of the rooted ones. As such, it can be seen as a kind of integration, since vertex rooting $F \mapsto F^\bullet$ of species satisfies

$$F^\bullet(X) = X F'(X).$$

**Proposition 2** Dissymmetry Theorem ([1] (4.2.46)). We have the following isomorphism of species:

$$E_2(C_M^\bullet) + \mathcal{M}(X E(C_M^\bullet)) = C_M + (C_M^\bullet)^2,$$

where $E_2$ denotes the species of two-element sets.

**Proof.** To any $C_M$-graph $g$, we can associate the auxiliary bipartite lump-bridge incidence graph $lb(g)$. This graph is actually a tree and we define the center of $g$ as
the center (lump or bridge) of \( \text{lb}(g) \). Let us use the exponents \( b, \ell, \) and \( b\ell \) to denote the rooting of a \( C_M \)-graph at one of its bridge, lump or pair of incident bridge-lump, respectively. Then the equation (2) can be expressed as

\[
C^\ell_M + C^b_M = C_M + C^{b\ell}_M,
\]

as is easily verified. Now the left-hand-side represents the species of \( C_M \)-graphs which have been rooted at either a bridge or a lump (a cell). It can happen that the rooting has been performed right at the center. This is canonically equivalent to doing nothing and is represented by the term \( C_M \) in the right-hand-side. On the other hand, if the rooting is done at an off-center cell, a bridge or a lump, then there is a unique incident cell of the other kind (a lump for a bridge and vice-versa) which is located towards the center, thus defining a unique \( C^{b\ell}_M \)-structure. It is easily checked that this correspondence is bijective and independent of any labelling, giving the desired species isomorphism.

Now let \( \mathcal{M}(X, Z) = \mathcal{M}(XE(Z)) \) be the two-sort species of graphs obtained by attaching an arbitrary number of external leaves (or legs if one prefers) at each vertex of an \( \mathcal{M} \)-graph. The legs belong to the second sort, denoted by \( Z \). We define the two-sort species \( V(X, Z) \) by

\[
V(X, Z) = aE_2(Z) - \mathcal{M}(X, Z).
\]

The term \( aE_2(Z) \) is to be interpreted as two legs attached together by an edge with weight \( a \). Here \( a \) is a formal variable. Note that

\[
\mathcal{M}^\times(X, Z) := X \frac{\partial}{\partial X} \mathcal{M}(X, Z) = \frac{\partial}{\partial Z} \mathcal{M}(X, Z)
\]

so that

\[
\frac{\partial}{\partial Z} V(X, Z) = aZ - \mathcal{M}^\times(X, Z).
\]

A similar definition is given for the two-sort species \( \mathcal{G}(X, T) \) which will turn out to be the Legendre transform of \( V(X, Z) \). The sort of legs will now be represented by \( T \). We set

\[
\mathcal{G}(X, T) = bE_2(T) + C_{M,b}(X, T),
\]

where \( C_{M,b}(X, T) = C_M(XE(T)) \), with the variable \( b \) acting as a bridge counter. By convention, a \( bE_2(T) \)-structure (two legs attached with an edge, of weight \( b \)) is also considered as a \( C_M \)-graph. We have

\[
\frac{\partial}{\partial T} \mathcal{G}(X, T) = bT + bC^\times_{M,b}(X, T).
\]
This species can be identified as that of \textit{planted} (with a \(b\)-weighted half bridge) \(C_{M+b}\)-graphs. We denote it by \(\mathcal{A}(X, T)\). The fundamental relation (1) can then be restated in terms of \(C_{M}\)-graphs with legs, as follows:

\textbf{Proposition 3} The species

\[
\mathcal{A}(X, T) := \frac{\partial}{\partial T} \mathcal{G}(X, T) = bT + bC_{M,b}^{\mathcal{X}}(X, T)
\]

satisfies the functional equation

\[
\mathcal{A}(X, T) = bT + b\mathcal{M}^{\mathcal{X}}(X, \mathcal{A}(X, T)).
\]

Also, the Dissymmetry Theorem takes the following form for \(C_{M}\)-graphs with legs and bridge counter \(b\).

\textbf{Proposition 4} We have the following isomorphism of species:

\[
\mathcal{G}^b(X, T) + \mathcal{G}'(X, T) = \mathcal{G}(X, T) + \mathcal{G}^{b\mathcal{L}}(X, T),
\]

which is equivalently expressed as

\[
\frac{1}{b} E_2(\mathcal{A}) + \mathcal{M}(X, \mathcal{A}) = \mathcal{G}(X, T) + \frac{1}{b} \mathcal{A}(\mathcal{A} - bT),
\]

with \(\mathcal{A} = \mathcal{A}(X, T)\).

The reader is invited to check by himself the term by term equivalence of (10) and (11).

\section{Legendre transform}

In the standard context of convex functions, the Legendre transform \(F(t)\) of a function \(V(z)\) is defined by

\[
F(t) = \sup_z (tz - V(z)).
\]

In the context of formal power series, we interpret instead the supremum to mean that \(z\) should be replaced by the formal power series solution \(z = z(t)\) of \(t = V'(z)\), assuming that \(V''(0) \neq 0\). This definition can be lifted to bivariate series \(V(x, z)\) and \(F(x, t)\) and also to bisort species, as follows.
**Definition.** Let $V(X, Z)$ be a two-sort species such that

$$
\frac{\partial}{\partial Z} V(0, 0) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial Z^2} V(0, 0) \neq 0.
$$

(12)

Then the Legendre transform of $V(X, Z)$ with respect to $Z$ is the two-sort species $F(X, T)$ defined by

$$
F(X, T) = \left(TZ - V(X, Z)\right)\bigg|_{Z=A(X,T)},
$$

(13)

where $A(X, T)$ is the compositional inverse, with respect to $Z$ of $\frac{\partial}{\partial Z} V(X, Z)$. In other words, we should have

$$
T = \frac{\partial}{\partial Z} V(X, A(X, T)).
$$

(14)

Note that the conditions (12), ensure the existence of the species $A(X, T)$. Also, a simple computation shows that

$$
\frac{\partial}{\partial T} F(X, T) = A(X, T).
$$

(15)

We want to apply this construction to line-irreducible graphs. For technical reasons, we replace $aE_2(Z)$ by $aZ^2 - aE_2(Z)$ in the definition (4) of $V(X, Z)$. These two species have the same exponential generating series, $az^2/2$, and the same partial derivative with respect to $Z$, namely $aZ$. Hence, we set

$$
V(X, Z) = aZ^2 - aE_2(Z) - M(X, Z).
$$

(16)

In this case, using (5), the functional equation (14) gives

$$
T = aA(X, T) - M^{\bullet x}(X, A(X, T))
$$

(17)

or, equivalently,

$$
A(X, T) = \frac{1}{a} T + \frac{1}{a} M^{\bullet x}(X, A(X, T)).
$$

(18)

This is precisely the functional equation satisfied by the species $A = A(X, T)$ of planted $C_{M,b}$-graphs, with $b = 1/a$. See Proposition 3.

**Theorem 5** Let $\mathcal{M}$ be a class of line-irreducible graphs, and let $V(X, Z)$ and $G(X, T)$ be the two-sort species defined by equations (16) and (6), respectively, and assume that $ab = 1$. Then $G(X, T)$ is equal to the Legendre transform $F(X, T)$ of $V(X, Z)$ with respect to $Z$.
**Proof.** We already know that $G(X, T)$ and $F(X, T)$ have the same partial derivative with respect to $T$, namely $A(X, T)$, and that $A(X, T)$ is the $Z$-compositional inverse of $V(X, Z)$. However, given the multiplicity of primitives that a species can have (see [4]), we still have to check that $G(X, T)$ coincides with $F(X, T)$ as defined in (13). This is achieved with the help of the Dissymmetry Theorem. Indeed, using (11), we have, with $A = A(X, T)$,

\[
G(X, T) = \frac{1}{b} E_2(A) + M(X, A) - \frac{1}{b} A(A - bT) \\
= T A - (aA^2 - aE_2(A) - M(X, A)) \\
= T A - V(X, A) \\
= F(X, T).
\]

This completes the proof. \qed

**References**


