2-CONNECTED GRAPHS WITH PRESCRIBED
THREE-CONNECTED COMPONENTS

by

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**Introduction**

Our goal:

1. An analogue of the block-cutpoint tree $bc(g)$ of a connected graph $g$, for 2-connected graphs

2. Functional equations for 2-connected graphs and networks with prescribed 3-connected components

3. Enumerative consequences and applications

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**Figure 1:** A 2-connected graph
Figure 2: A separating pair

Figure 3: Bond along a separating pair
Figure 4: Separating pairs
Figure 5: Separating pairs and tricomponents

**Tricomponents:** 3-connected graphs \((A, B,\) and \(C)\) or polygons: \((S, T, U,\) and \(P)\).

Figure 6: tc-tree of a 2-connected graph
Dissymmetry Theorem for 2-connected graphs

Let $\mathcal{F}$ be a given class of 3-connected graphs.

$\mathcal{B} = \mathcal{B}_\mathcal{F}$ denotes the class of 2-connected graphs all of whose 3-connected components are in $\mathcal{F}$. Note: $K_2$ is in $\mathcal{B}$.

We introduce the following classes of *rooted* graphs in $\mathcal{B}$:

- $\mathcal{B}^\circ$: with a distinguished tricomponent;
- $\mathcal{B}^\bullet$: with a distinguished bond;
- $\mathcal{B}^{\circ\bullet}$: with a distinguished pair of adjacent tricomponent and bond.

**Theorem.** We have the following identity (species isomorphism):

$$\mathcal{B}^\circ + \mathcal{B}^\bullet = \mathcal{B} - K_2 + \mathcal{B}^{\circ\bullet}.$$

**Proof.** Analyse the relative positions of the rooted tricomponents or bonds with respect to the graph "centers".
2-pole networks

A 2-pole network (or simply a network) is a connected graph \( N \) with two distinguished vertices 0 and 1, such that the graph \( N \cup 01 \) is unseparable.

The vertices 0 and 1 are called the poles of \( N \), and all the other vertices of \( N \) are said to be internal. The internal vertices of a network form its underlying set.

Figure 7: (i) series-parallel network (ii) \((K_4)_{0,1}\) - network

Figure 8: The trivial networks: (i) \( \parallel \) (ii) \( y \parallel \)
For any class of graphs $\mathcal{G}$, we define an associated class of networks $\mathcal{G}_{0,1}$. A network in $\mathcal{G}_{0,1}$ is obtained from a graph in $\mathcal{G}$ by selecting and removing an edge and relabelling its endpoints with 0 and 1.

For memory, the corresponding exponential generating functions satisfy

$$x^2 \mathcal{G}_{0,1}(x, y) = 2 \frac{\partial}{\partial y} \mathcal{G}(x, y)$$

We define an operator $\tau$ acting on 2-pole networks, $N \leftrightarrow \tau \cdot N$, which interchanges the poles 0 and 1. A class $\mathcal{N}$ of networks is called symmetric if $N \in \mathcal{N} \implies \tau \cdot N \in \mathcal{N}$.
The composition $\mathcal{M} \uparrow \mathcal{N}$

Let $\mathcal{M}$ be a class of graphs (or networks) and $\mathcal{N}$ be a symmetric class of networks. We denote by $\mathcal{M} \uparrow \mathcal{N}$ the class of pairs of graphs (or networks) $(M, T)$, such that

1. the graph (or network) $M$ (called the core) is in $\mathcal{M}$,
2. the vertex set $V(M)$ is a subset of $V(T)$,
3. there exists a family $\{N_e\}$ of networks in $\mathcal{N}$ (called the components) such that the graph $T$ can be obtained from $M$ by substituting $N_e$ for each edge $e$ of $M$, the poles of $N_e$ being identified with the extremities of $e$.

Figure 10: Example of a $(\mathcal{M} \uparrow \mathcal{N})$-structure $(M, T)$
Examples of compositions

1. If we take the class $\mathcal{G} = \{K_2\}$ for cores and the class $\mathcal{N}$ of all networks for components, then the $(K_2 \uparrow \mathcal{N})$-structures consist of graphs $G$ together with two selected (adjacent or not) vertices $a$ and $b$, such that the graph $G \cup ab$ is 2-connected.

2. The composition $\mathcal{M} \uparrow \mathcal{N}$ is called canonical if for any structure $(M, T) \in \mathcal{M} \uparrow \mathcal{N}$, the core $M \in \mathcal{M}$ is uniquely determined by the graph (or network) $T$. In this case, we can identify $\mathcal{M} \uparrow \mathcal{N}$ with the class of resulting networks $T$.

For example, we can take $\mathcal{G} = K$, the class of complete graphs, $\mathcal{N} = \mathbb{1} + y\mathbb{1}$, the class of trivial networks, (see Figure 4). Then we have

$$K \uparrow (\mathbb{1} + y\mathbb{1}) = \mathcal{G},$$

where $\mathcal{G}$ denotes the class of all graphs, the composition being canonical.
Let \( \mathcal{F} \) be a given class of 3-connected graphs. 

\( \mathcal{B} = \mathcal{B}_{\mathcal{F}} \) denotes the class of 2-connected graphs all of whose 3-connected components are in \( \mathcal{F} \).

\( \mathcal{D} = \mathcal{D}_{\mathcal{F}} \) denotes the class of networks all of whose 3-connected components are in \( \mathcal{F} \).

In fact \( \mathcal{D} = (1 + y)\mathcal{B}_{0,1} - \mathbb{1} \).

**Theorem.** We have the following identities:

\[
\mathcal{B}^\circ = \mathcal{F} \uparrow \mathcal{D} + \mathcal{C} \uparrow (\mathcal{D} - \mathcal{S})
\]

\[
\mathcal{B}^\bullet = K_2 \uparrow \left( (1 + y)E_{\geq 2}(\mathcal{H} + \mathcal{S}) - E_2(\mathcal{S}) \right)
\]

\[
\mathcal{B}^{\circ-\bullet} = K_2 \uparrow \left( (1 + y)(\mathcal{H} + \mathcal{S})E_{\geq 1}(\mathcal{H} + \mathcal{S}) - \mathcal{S}^2 \right)
\]

where

\( \mathcal{C} \) is the class of unoriented cycles of length \( \geq 3 \) (polygons),

\( \mathcal{S} \) is the class of (strictly) series networks,

\( \mathcal{H} = \mathcal{F}_{0,1} \uparrow \mathcal{D} \) (irreducible networks).
Theorem. (Trakhtenbrot)

The species of two-pole networks $\mathcal{D}$ corresponding to a species of 3-connected graphs $\mathcal{F}$ can be expressed by the relation

$$\mathcal{D} = (1 + y)E(\mathcal{F}_{0,1} \uparrow \mathcal{D} + \frac{X\mathcal{D}^2}{1 + XD}) - 1.$$ 

Proof. It follows from the tricomponent decomposition of a 2-connected graph that

$$\mathcal{D} = y1 + (1 + y)(\mathcal{H} + \mathcal{S} + E_{\geq 2}(\mathcal{H} + \mathcal{S}))$$
$$= (1 + y)E(\mathcal{H} + \mathcal{S}) - 1,$$

where

$\mathcal{H} = \mathcal{F}_{0,1} \uparrow \mathcal{D}$ and $\mathcal{S}$ satisfies

$$\mathcal{S} = (\mathcal{D} - \mathcal{S}) \cdot_s \mathcal{D} = (\mathcal{D} - \mathcal{S})XD$$

and hence

$$\mathcal{S} = \frac{X\mathcal{D}^2}{1 + XD}.$$ 

Corollary. In terms of the exponential generating functions we have

$$\mathcal{D}(x, y) = (1 + y)\exp \left( \mathcal{F}_{0,1}(x, \mathcal{D}(x, y)) + \frac{x\mathcal{D}^2(x, y)}{1 + x\mathcal{D}(x, y)} \right) - 1.$$
References


